ISRAEL JOURNAL OF MATHEMATICS, Vol. 62, No. 2, 1988

SUCCESSORS OF SINGULARS, COFINALITIES OF REDUCED PRODUCTS OF CARDINALS AND PRODUCTIVITY OF CHAIN CONDITIONS

BY

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ABSTRACT

We continue here our investigation of cofinalities of reduced products of regular cardinals and give some applications, such as the non-productiveness of λ^+-c.c. when $\lambda > 2^{\mathrm{cf}\,\lambda}$.

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§1. Introduction

A Boolean algebra B satisfies the λ-chain condition (λ-c.c.) if any family of pairwise disjoint nonzero members of B has cardinality $< \lambda$. For a topological space X, by $c(X)$ we denote the cellularity of X which is the supremum of

† Research partially supported by the United States–Israel Binational Science Foundation.
Received June 7, 1987 and in revised form December 31, 1987

cardinalities of families of pairwise disjoint nonempty open sets in X. Clearly $c(X) \leqq \lambda$ iff the regular open algebra of X satisfies λ^+-c.c. The problem whether the λ-c.c. is a productive property (of Boolean algebras or, equivalently, of topological spaces) has a very long history which we do not repeat here (see, for example, [T1], [T2]; Juhasz [J] and Monk [M] stress this problem in preliminary versions of their surveys). First ZFC examples of regular cardinals λ for which λ-c.c. is not productive were given by Todorcevic [T1]. The cardinals of [T1] are of the form of 2^κ where $\operatorname{ded}(\kappa, 2^\kappa)$ holds (e.g., $\kappa = \aleph_0$), so all of them might be weakly inaccessible. Hence [T1] didn't solve the topological question: Is always $c(X \times Y) \leqq c(X) \cdot c(Y)$? Note that this question is trivially equivalent to the question: Is λ^+-c.c. productive for all λ? In [T2], Todorcevic answered this question (in ZFC) by providing a class of cardinals λ for which λ^+-c.c. is not productive: the λ such that $(\forall \mu < \lambda)(\mu^{\operatorname{cf}\lambda} < \lambda)$. Todorcevic [T2] uses (e.g., when $\lambda = (2^{\aleph_0})^{+\omega}$) [Sh 1] about cofinalities of reduced products of regular cardinals. He also got negative partition relations, e.g., $\lambda^+ \nrightarrow [\lambda^+]^2_{\operatorname{cf}\lambda}$ when $(\forall \mu < \lambda)[\mu^{\operatorname{cf}\lambda} < \lambda]$, and got $\lambda - S$ and $\lambda - L$ spaces. He told me that a proof of the consistency of "λ^+-c.c. is productive" will be the real generalization of MA (unlike some soft ones; see e.g. [Sh 2]). By [Sh 7], this fails for $\lambda \geqq 2^{\aleph_0}$ regular (even $\operatorname{Pr}_0(\lambda^+, \aleph_0, \aleph_0)$ fails, see Definition 31 and for some consequences 36, 37). This makes it more desirable to get parallel results not just for some successor of singular, but for quite many.

In [Sh 1], [Sh 3], [Sh 4], [Sh 5, Ch. XIII, §5, §6] and [M Sh] the cofinalities of reduced products of regular cardinals (usually $\leqq \kappa$ cardinals $> 2^\kappa$) were investigated for various purposes. We continue this here (but start from the beginning, for the reader's convenience, giving the statements but usually not the proof of the repeated parts, but in a way that the reader may reconstruct them).

The main new conclusions are (the hypothesis can be weakened, and apply to some weakly inaccessible):

A. THEOREM. *If* $\lambda = \mu^+$, $\mu > 2^\kappa$, $\kappa = \operatorname{cf}\mu$, *or* $\lambda = \operatorname{cf}(\Pi_{1<\kappa}\lambda_i/\mathscr{D})$ *where* $2^\kappa > \lambda$, $\lambda_i < \lambda$, $\mathscr{D}$ *a filter on* κ, *then*:

(A) *for some regular cardinals* σ *and* λ_i *from* $((2^\kappa)^+, \lambda)$ *for* $i < \sigma$, $\sigma \leqq \kappa$ $(i < j \Rightarrow \lambda_i < \lambda_j)$ *and letting* $\mathscr{D}^{\mathrm{cobd}}_\sigma = \{a \subseteq \sigma : \sigma - a \text{ bounded}\}$, $\Pi_{i<\sigma}\lambda_i/\mathscr{D}^{\mathrm{cobd}}_\sigma$ *has true cofinality* λ, *i.e., there are* $f_\alpha \in \Pi_{i<\sigma}\lambda_i$ *for* $\alpha < \lambda$ $[\alpha < \beta \Rightarrow f_\alpha/\mathscr{D} < f_\beta/\mathscr{D}]$ *and*

$$\left(\forall f \in \prod_{i<\sigma} \lambda_i\right)(\exists \alpha < \lambda^+)[f/\mathscr{D}^{\mathrm{cobd}} < f_\alpha/\mathscr{D}^{\mathrm{cobd}}];$$

(B) *there are regular cardinals* $\lambda_i \in (2^\kappa, \lambda)$ *for* $i < \delta$, $\delta < \kappa^+$ $[i < j \Rightarrow \lambda_i < \lambda_j]$ *and a* $|\delta|$*-complete filter* $\mathscr{D}$ *on* δ, *and* $\langle f_\alpha : \alpha < \delta \rangle$ *witnessing* $\Pi_{i<\delta} \lambda_i / \mathscr{D}$ *has true cofinality* λ, *which form a kind of a strong Kurepa tree*: *i.e., for each* i, $\{f_\alpha \restriction i : \alpha < \lambda\}$ *has cardinality* $< \lambda_i$.

We then define some strong negative partition relations, observe their interrelations and consequences, and draw some of them from the work on the cofinalities (mainly Theorem I). Part (B) of Theorem I is motivated by its being used by Todorcevic [T2]. In that case it follows by cardinal arithmetic. Lastly we deal with powers of singulars, giving some information of $(\lambda, 2^\lambda)$. We may represent the proof of $\aleph_\delta^{\mathrm{cf}\,\delta} < \aleph_{(|\delta|^{\mathrm{cf}\,\delta})^+}$ in a simpler way than in [Sh 5], at least in most cases.[1]

Conventions and notations

$\alpha, \beta, \gamma, \zeta, \xi, i, j, \delta$ are ordinals, δ should be limit;
$\lambda, \kappa, \mu, \chi, \theta, \sigma$ are cardinals, usually infinite;
for a set a, $\Pi a = \{f : \mathrm{dom}\, f = a, f(\theta) \in \theta \text{ for } \theta \in a\}$; $\mathscr{P}(a) = \{b : b \subseteq a\}$;
for a sequence $\bar\lambda = \langle \lambda_i : i \in u \rangle$, $\Pi_{i \in u} \lambda_i = \Pi \bar\lambda = \{f : \mathrm{dom}\, f = u, f(i) \in \lambda_i\}$;
for a filter $\mathscr{D}$ on u, $\mathscr{D} + A = \{X \subseteq u : X \cup (u - A) \in \mathscr{D}\}$;
for an ideal I on u, $I + A = \{X \subseteq u : X - A \in I\}$.
Many times we do not distinguish between a filter and the ideal dual to it.
$\mathscr{D}_\lambda$ is the filter of closed unbounded subsets of λ, for λ regular uncountable.
$\mathscr{D}_\lambda^{\mathrm{cobd}}$ is the filter $\{A : A \subseteq \lambda, |\lambda - A| < \lambda\}$, $J_a^{\mathrm{bd}} = \{b \subseteq a : |b| < |a|\}$. If $(P, \leqq)$ is a partial (quasi)order, $p < q$ means $p \leqq q \wedge \neg q \leqq p$.

§1. Cofinalities

1. Definition. For a partial order P, $p \in P$ is a lub (a least upper bound) of $A \subseteq P$ *if* it is an upper bound $((\forall x \in A)(x \leqq p))$ and $p \leqq q$ for any upper bound q of A.

We say $p \in P$ is a weak lub of A if it is an upper bound and no $q < p$ is an upper bound.

1A. Fact. (1) If a lub exists, it is unique up to equivalence.

[1] For further advances, see [Sh 11] on λ^+-c.c is not productive, e.g. for λ regular $> \aleph_1$; and [Sh 12] for more information on cofinalities of products of regular cardinals.

(2) If $A, B \subseteq P$, $\forall p \in A\ \exists q \in B (p \leqq q)$, $\forall q \in B\ \exists p \in A (q \leqq p)$, *then* A, B has the same lub (if any).

1B. NOTATION. (1) $R^{\kappa,0} = \{\lambda : \lambda \text{ a regular cardinal}\}$,
$R^{\kappa,1} = \{\lambda \in R^{\kappa,0} : \lambda > \kappa\}$,
$R^{\kappa,2} = \{\lambda \in R^{\kappa,0} : \lambda > 2^\kappa\}$,
$I^{\kappa,l} = \{a \subseteq R^{\kappa,l} : |a| \leqq \kappa\}$,
$\mathscr{I}^{\kappa,l} = \{\bar\lambda : \bar\lambda = \langle \lambda_i : i \in u \rangle$ a sequence of cardinals from $R^{\kappa,l}$, $|u| \leqq \kappa\}$.

(2) If we omit l it means $l = 2$; we usually omit κ as it is fixed.

2. DEFINITION. For a partial order P (which is $(|P|, <)$ or $(|P|, \leqq)$):

(a) $\mathrm{cf}(P) = \mathrm{Min}\{|A| : A \subseteq P, (\forall p \in P)(\exists q \in A) p \leqq q\}$ (it is always well defined). Such A is called a cover.

(b) λ is a possible scale of P if there are $p_i \in P$ for $i < \lambda$ such that $[i < j < \lambda \Rightarrow p_i < p_j]$, $(\forall p \in P) \bigvee_i p_i \nleqq p$ and λ is a regular cardinal.

(c) $\mathrm{psc}(P) = \{\lambda : \lambda \text{ is a possible scale of } P\}$.

(d) λ is the true cofinality of P, $\lambda = \mathrm{tcf}(P)$ if it is a regular cardinal and there are $p_i \in P$ for $i < \lambda$, $[i < j < \lambda \Rightarrow p_i < p_j]$ and $(\forall q \in P) \bigvee_i q \leqq p_i$. [Note that $\mathrm{tcf}(P)$, if it exists, is unique and then $\mathrm{cf}(P) = \mathrm{tcf}(P)$, $\mathrm{psc}(P) = \{\mathrm{tcf}(P)\}$.] Also $\mathrm{tcf}(P) = 1$ if there is $p \in P$ such that $(\forall q \in P)\ q \leqq p$.[2]

3. DEFINITION. (1) For $\bar\lambda = \langle \lambda_i : i \in u \rangle$ a sequence of cardinals, and $\mathscr{D}$ a filter on u, *then* for $f, g \in \prod_{i \in u} \lambda_i$, $f \leqq g \bmod \mathscr{D}$, or $f \leqq_{\mathscr{D}} g$ iff $\{i \in u : f(i) \leqq g(i)\} \in \mathscr{D}$ [it is a partial order].

(2) If $\mathscr{D} = \{u\}$, we omit it.

(3) If $\mathscr{D}$ is a filter on some u', $u \subseteq u'$, this will mean

$$\{i \in u : f(i) \leqq g(i)\} \cup (u' - u) \in \mathscr{D}.$$

(4) We sometimes replace $\mathscr{D}$ by the dual ideal.

(5) We can replace $\leqq$ by any other relation (in particular $<$).

4. FACT. (1) If $\mathrm{cf}\,\alpha_i = \mathrm{cf}\,\beta_i$ for $i \in u$, $\mathscr{D}$ a filter on u, then:

(a) $\mathrm{cf}(\prod \alpha_i, \leqq_{\mathscr{D}}) = \mathrm{cf}(\prod \beta_i, \leqq_{\mathscr{D}})$;

(b) $\mathrm{tcf}(\prod \alpha_i, \leqq_{\mathscr{D}}) = \lambda \Leftrightarrow \mathrm{tcf}(\prod \beta_i, \leqq_{\mathscr{D}}) = \lambda$;

(c) $\mathrm{psc}(\prod \alpha_i, \leqq_{\mathscr{D}}) = \mathrm{psc}(\prod \beta_i, \leqq_{\mathscr{D}})$;

[2] Remember, if we are using $P = (P, \leqq)$ then $p < q$ means $p \leqq q$ or $p = q$.

(d) if in addition $u(*)=\{i\in u:\alpha_i$ is limit$)\neq\emptyset \bmod \mathscr{D}$, *then* for $X\in\{\mathrm{cf},\mathrm{tcf}\}$

$$X\left(\prod_{i\in u}\alpha_i,\ \leqq_{\mathscr{D}}\right)=X\left(\prod_{i\in u}\alpha_i,\ \leqq_{\mathscr{D}+u(*)}\right);^{(3)}$$

(e) if $\{i\in u:\alpha_i$ is limit$\}\in\mathscr{D}$ and $\langle f_\alpha: <\lambda\rangle$ witnesses $\lambda=\mathrm{tcf}(\Pi_{i\in u}\alpha_i,\ \leqq_{\mathscr{D}})$ *then* for some club C of λ

$$\alpha\in C\wedge\beta\in C\wedge\alpha<\beta\Rightarrow f_\alpha<_{\mathscr{D}}f_p$$

(so wlog $C=\lambda$).

(2) If $\mathscr{D}$ is an ultrafilter over u then $(\Pi_{i\in u}\lambda_i,\ \leqq_{\mathscr{D}})$ has a true cofinality (in fact every P such that $(\forall x,y\in P)(x\leqq y\vee y\leqq x)$).

(3) Suppose for a filter $\mathscr{D}$ on u,

$$\mathrm{tcf}\left(\prod_{i\in u}\lambda_i,\ \leqq_{\mathscr{D}}\right)=\lambda,\quad \bigwedge_{i\in u}|u|<\lambda_i\,\mathrm{cf}\,\lambda_i\neq\lambda\quad\text{and } |u|<\lambda,$$

then

(a) for each cardinal μ,

$$\{i:\lambda_i=\mu\}=\emptyset\ \bmod \mathscr{D},\text{ and }\lambda\geqq\mathrm{Min}\{\kappa:\{i:\lambda_i<\kappa\}\in\mathscr{D}\}$$

and

(b) for each filter $\mathscr{D}_1$ extending $\mathscr{D}$, $\mathrm{tcf}(\Pi\lambda_i,\ \leqq_{\mathscr{D}_1})=\lambda$ (here the assumptions on u, λ_i are not necessary, except "λ_i a limit ordinal").

Also

(c) defining $[iEj\Leftrightarrow i\in u\wedge\lambda_i=\lambda_j]$ (an equivalence relation) $u/E=\{i/E:i\in u\}$, $\mathscr{D}/E=\{A\subseteq u/E:\bigcup\{i/E:i/E\in A\}$ belong to $\mathscr{D}\}$, $\lambda_{i/E}=\lambda_i$, we have

$$\mathrm{tcf}\left(\prod_{\zeta\in u/E}\lambda_\zeta,\ <_{\mathscr{D}/E}\right)=\lambda.$$

(4) In (3)(c), if $\mathscr{D}$ is σ-complete then $\mathscr{D}/E$ is σ-complete; if

$$\mathscr{D}=\{v:v\subseteq u,\ |u-v|<\kappa\}$$

(3) Also $\mathrm{psc}(\Pi_{i\in u}\alpha_i,\leqq_{\mathscr{D}})=\mathrm{psc}(\Pi_{i\in u}\alpha_i,\leqq_{\mathscr{D}+u(*)})-\{1\}$ and $\{1\}\in\mathrm{psc}(\Pi_{i\in u}\alpha_i,\leqq_{\mathscr{D}})$ *iff* $u-u(*)\neq\emptyset$ mod $\mathscr{D}+u(*)$.

where $\kappa \leqq |u|$ or $u = \theta \in R^{\kappa,0}$, $\mathscr{D}$ normal,[4] then for some $s \subseteq u$, $s \neq \emptyset$ mod $\mathscr{D}$, and E is equality on s.

(5) If $\mathscr{D}$ is a filter on u,

$$\operatorname{tcf}\left(\prod_{i\in u} \lambda_i, \leqq_{\mathscr{D}}\right) = \lambda, \quad \bigwedge_{i\in u} \lambda_i < \lambda \quad \text{and} \quad 2^{|u|} < \lambda$$

then $\{i \in u : \lambda_i \leqq |u|\} = \emptyset$ mod $\mathscr{D}$ so the conclusion of (3) holds.

(6) In (5), if $\lambda > \mu^{|u|}$ (e.g. $\mu = 2^{|u|}$) then $\{i \in u : \lambda_i \leqq \mu\} = \emptyset$ mod $\mathscr{D}$, so $\operatorname{tcf}(\Pi\{\lambda_i : \lambda_i > \mu\}, \leqq_{\mathscr{D}}) = \lambda$.

(7) If $f_\alpha \in \Pi_{i\in u} \lambda_i$ for $\alpha < \delta$, $\mathscr{D}$ a filter on u, $u^* \in \mathscr{D}$, $\operatorname{cf}\delta > |u|$ (or alternatively, $\mathscr{D}$ is $(\operatorname{cf}\delta)^+$-complete) and $[\alpha < \beta < \delta \wedge i \in u^* \Rightarrow f_\alpha(i) \leqq f_\beta(i)]$, *then* $\{f_\alpha : \alpha < \delta\}$ has a $\leqq_{\mathscr{D}}$-lub f, where

$$f(i) = \begin{cases} \bigcup_{\alpha<\delta} f_\alpha(i), & i \in u^*, \\ 0, & i \notin u^*. \end{cases}$$

(8) If λ is regular $> 2^{|u|}$, $\mathscr{D}$ a filter on u, $\langle f_\alpha : \alpha \leqq \lambda \rangle$ is $\leqq_{\mathscr{D}}$-increasing, *then* for some $v \subseteq u$:

(i) if $v \neq \emptyset$ mod $\mathscr{D}$ then $\langle f_\alpha/(\mathscr{D} + v) : \alpha < \lambda \rangle$ is eventually constant,

(ii) if $u - v \neq \emptyset$ mod $\mathscr{D}$ then for some club $C \subseteq \lambda$, $\langle f_\alpha : \alpha \in C \rangle$ is $<_{\mathscr{D}}$-increasing.

(9) If $\mathscr{D}$ is a filter on u, λ regular, $f_\alpha \in \Pi_{i\in u} \lambda_i$ for $\alpha < \lambda$, $[\alpha < \beta \Rightarrow f_\alpha \leqq_{\mathscr{D}} f_\beta]$ and in $\mathscr{P}(u)/\mathscr{D}$ there is no strictly increasing sequence of elements of length λ, *then* the conclusion of (8) above holds [use the assumption twice].

(10) Suppose λ is regular $> |u|$, $\mathscr{D}$ a filter on u, $\langle f_\alpha : \alpha < \lambda \rangle$ is $\leqq_{\mathscr{D}}$-increasing and $\langle f_\alpha/\mathscr{D} : \alpha < \lambda \rangle$ is not eventually constant. Let

$$E = \{v \subseteq u : \langle f_\alpha/(\mathscr{D} + (u - v)) : \alpha < \lambda \rangle \text{ is eventually constant}\}.$$

Then

(i) $\langle f_\alpha : \alpha < \lambda \rangle$ is $\leqq_E$-increasing;

(ii) if $v \neq \emptyset$ mod E then $\langle f_\alpha/(\mathscr{D} + v) : \alpha < \lambda \rangle$ is not eventually constant;

(iii) if $\mathscr{D}$ is σ-complete then E is σ-complete (and always E is a proper filter).

5. Fact. (1) Suppose $\mathscr{D}$ is a filter on u, $f_\alpha \in \Pi_{i\in u} \lambda_i$ for $\alpha < \lambda$, $f_\alpha \leqq f_\beta$ mod $\mathscr{D}$ for $\alpha < \beta < \lambda$, and λ is regular $> 2^{|u|}$. Then there is $f \in \Pi_{i\in u}(\lambda_i + 1)$ such that

(a) f is a $\leqq_{\mathscr{D}}$-least upper bound of $\langle f_\alpha : \alpha < \lambda \rangle$:

$$f_\alpha \leqq_{\mathscr{D}} f \qquad \text{for } \alpha < \lambda,$$

$$(\forall f')\left[\bigwedge_{\alpha<\lambda} f_\alpha \leqq_{\mathscr{D}} f' \Rightarrow f \leqq_{\mathscr{D}} f'\right];$$

(b) w.l.o.g. $(\forall i) f(i) \neq 0$ (provided that $\{i : f(i) = 0\} = \emptyset$ mod $\mathscr{D}$) and for every $f' <_{\mathscr{D}} f$, for some α, $f' <_{\mathscr{D}} f_\alpha$ (if $\{i : f(i) \text{ limit}\} \in \mathscr{D}$).

(2) If λ is regular $> 2^{|u|}$, $\mathscr{D}$ a filter on u, $\langle f_\alpha : \alpha \leqq \lambda \rangle$ is $<_{\mathscr{D}}$-increasing, *then* for some $f \leqq f_\lambda$, $\mathrm{tcf}(\Pi_{i\in u} f(i), <_{\mathscr{D}}) = \lambda$.

PROOF. See [Sh 4] 2.3.

6. DEFINITION. (1) For any $a \in I^{\kappa,l}$

(a) $\mathrm{pcf}(a) = \{\mathrm{cf}(\Pi a, \leqq_{\mathscr{D}}) : \mathscr{D} \text{ an ultrafilter over } a\}$.

(b) For a property Γ of filters,

$$\mathrm{pcf}_\Gamma(a) = \{\mathrm{tcf}(\Pi a, \leqq_{\mathscr{D}}) : \mathscr{D} \text{ a filter over } a, \vDash \Gamma(\mathscr{D})\}.$$

(c) For Γ = being σ-complete, we write σ; for Γ = "extending $\mathscr{D}$" we write $\mathscr{D}$.

(2) Similarly for $\bar{\lambda} \in \mathscr{I}^{\kappa,l}$..

6A. REMARK. (1) As the number of ultrafilters over a is $2^{2^{|a|}}$, we know that $|\mathrm{pcf}(a)| \leqq 2^{2^{|a|}}$; however (see 16(3)) $|\mathrm{pcf}(a)| \leqq 2^\kappa$.

(2) For $l = 2$ the difference between dealing with $I^{\kappa,l}$, and $\mathscr{I}^{\kappa,l}$ is slight by 4(2)–4(6). For $l = 0, 1$ $I^{\kappa,l}$ is more interesting.

6B. PROBLEM. Can $|\mathrm{pcf}(a)| > |a|$ for $a \in I^{\kappa,2}$?

7. FACT. $\mathrm{pcf}_{\aleph_0}(a) = \mathrm{pcf}(a)$ (for any $a \in I^{\kappa,l}$).

8. DEFINITION. For a regular λ (we may omit l if $l = 2$):

(1) $I^{\kappa,l}_{<\lambda} = \{a \in I^{\kappa,l} : \mathrm{cf}(\Pi a, \leqq) < \lambda\}$.

(2) $J^{a,l}_{<\lambda} = J^{\kappa,a,l}_{<\lambda} = \{b \subseteq a : b \in I^{\kappa,l}_{<\lambda}\}$ for $a \in I^{\kappa,l}_{<\lambda}$.

(3) $\mathscr{I}^{\kappa,l}_{<\lambda} = \{\bar{\lambda} \in \mathscr{I}^{\kappa,l} : \mathrm{cf}(\Pi\bar{\lambda}, \leqq) < \lambda\}$.

(4) $\mathscr{I}^{\bar{\lambda},l}_{<\lambda} = \mathscr{I}^{\kappa,\bar{\lambda},l}_{<\lambda} = \{u : u \subseteq \mathrm{dom}(\bar{\lambda}), \bar{\lambda} \restriction u \in \mathscr{I}^{\kappa,l}_{<\lambda}\}$ for $\bar{\lambda} \in \mathscr{I}^{\kappa,l}_{<\lambda}$ [so $u \in \mathscr{I}^{\bar{\lambda}}_{<\lambda}$ iff $u \subseteq \mathrm{dom}(\bar{\lambda})$ and $\mathrm{cf}(\bar{\lambda} \restriction u, \leqq) < \lambda$].

9. LEMMA. (1) *Suppose* $\lambda \in \mathrm{psc}(\Pi_{i\in u} \lambda_i, \leqq_I)$, *I an ideal on u, and*: $\lambda > 2^{|u|}$ *or at least in* $\mathscr{P}(u)/I$ *there is no strictly increasing sequence of elements of length* λ.

Then *there is an ideal* I_1 *on u such that*:

(i) $I \subseteq I_1$,

(ii) $\lambda = \mathrm{tcf}(\Pi_{i\in u} \lambda_i, \leqq_{I_1})$.

(2) *If I is σ-complete then there is a σ-complete ideal I_1 satisfying* (i) + (ii) *above.*

(3) *Similarly for normal filters.*

(4) *If $f_\alpha \in \Pi_{i \in u} \lambda_i$ for $\alpha < \lambda$, $[\alpha < \beta \Rightarrow f_\alpha <_I f_\beta]$ where I is an $\aleph_1$-complete ideal on u*, then *for some $g \in \Pi_{i \in u} (\lambda_i + 1)$ and $f'_\alpha = f_\alpha \bmod I$ for $\alpha < \lambda$, $f'_\alpha \in \Pi_{i \in u} g(i)$ and $\langle f'_\alpha : \alpha < \lambda \rangle$ is a witness for $\lambda \in \mathrm{psc}(\Pi_{i \in u} g(i), <_I)$.*

(5) *If $\langle \lambda_i : i < \delta \rangle$ is increasing, $\lambda_i \in R^{\kappa,0}$, $\delta < \lambda_0$ [or at least $\delta < \Sigma_{i<\delta} \lambda_i$], I an ideal on δ, δ is a limit ordinal, I includes all bounded subsets of δ*, then *in $\Pi_{i<\delta} \lambda_i$ there are f_α for $\alpha < (\Sigma_{i<\delta} \lambda_i)^+$ as in* (4).

(6) *If $\lambda \in \mathrm{psc}(\langle \lambda_i : i \in u \rangle, <_I)$, $\lambda > 2^{|u|}$ then there is a minimal I_1, satisfying* (i), (ii) *from* (1); *i.e. every I'_1 satisfying* (i), (ii) *from* (1) *extends I_1. Similarly for* (2), (3).

PROOF. (1) Let $\langle f_\alpha : \alpha < \lambda \rangle$ witness $\lambda \in \mathrm{psc}(\Pi_{i \in u} \lambda_i, \leqq_I)$. Define

$$I_1 = \{A \subseteq u : \langle f_\alpha : \alpha < \lambda \rangle \text{ does not witness } \lambda \in \mathrm{psc}(\Pi_{i \in u} \lambda_i, \leqq_{I+(u-A)})\}.$$

By 4(8) there are $v \leqq u$, C as there, so $v \in I_1$; and for notational simplicity let $C = \lambda$. So for $A \in I_1$, there is $g_A \in \Pi_{i \in u} \lambda_i$ such that $f_\alpha <_{I+(u-A)} g_A$ for $\alpha < \lambda$. Clearly $u \notin I_1$ (by the choice of $\langle f_\alpha : \alpha < \lambda \rangle$). Note that $I \subseteq I_1$ as $A \in I \Rightarrow u \in I + (u - A)$, hence $f_\alpha <_{I+(u-A)} f_0$ for every $\alpha < \lambda$ so $g_A \stackrel{\text{def}}{=} f_0$ is o.k.

Now I_1 is downward closed: if $A \in I_1$, $B \subseteq A$ then g_A witnesses that $B \in I_1$.

Next I_1 is closed under union: if $A, B \in I_1$, then g defined by $g(i) = \mathrm{Max}\{g_A(i), g_B(i)\}$ witness $A \cup B \in I_1$.

Lastly $\lambda = \mathrm{tcf}(\Pi_{i \in u} \lambda_i, I_1)$ because $\langle f_\alpha : \alpha < \lambda \rangle$ witness it; let us check the conditions of 2(d).

$f_\alpha \in \Pi_{i \in u} \lambda_i$ trivially; $\alpha < \beta \Rightarrow f_\alpha <_{I_1} f_\beta$ as $I \subseteq I_1$, $v \in I_1$, $C = \lambda$; lastly if $f \in \Pi_{i \in u} \lambda_i$, let $u_\alpha = \{i \in u : f(i) \geqq f_\alpha(i)\}$, *then* for some u^*, $\{\alpha < \lambda : u_\alpha = u^*\}$ is an unbounded subset of λ [or at least $\{\alpha < \lambda : u_\alpha = u^* \bmod I\}$ is an unbounded subset of λ], hence f can serve as g_{u^*} hence $u^* \in I_1$ and we finish easily.

(2), (3) Left to the reader.

(4) Choose by induction on n, $g_n \in \Pi_{i \in u} (\lambda_i + 1)$ such that $f_\alpha < g_n \bmod I$ for $\alpha < \lambda$, and $g_{n+1} < g_n \bmod I$.

Let g_0 be: $g_0(i) = \lambda_i$. As I is $\aleph_1$-complete, for some $n < \omega$, we have chosen g_n but cannot choose g_{n+1}. As $f_0 <_I g_n$, w.l.o.g. $(\forall i) g_n(i) > 0$. Let $f'_\alpha(i)$ be $f_\alpha(i)$ if $f_\alpha(i) < g_n(i)$ and be zero otherwise. Check.

(5) Left to the reader.

(6) We can choose for every $w \subseteq u$, I_1^w, $\langle f_\alpha^w : \alpha < \lambda \rangle$ as in the proof of (1) such that $w \notin I_1^w$, if possible. Easily $\{w \subseteq a : I_1^w \text{ is not defined}\}$ is as required.

10. Conclusion. (1) If

$$\lambda \in \mathrm{psc}\left(\prod_{i\in u} \lambda_i, \leqq_I\right), \quad \lambda_i < \lambda, \quad |u| < \lambda$$

(λ regular, I an ideal on u) and $\lambda > 2^{|u|}$ *or* at least in $\mathscr{P}(u)/I$ there is *no* strictly increasing sequence of length λ, *then* for some u_1, $|u_1| \leqq u$, regular cardinals μ_i ($i \in u_1$), $\mu_i \leqq \lambda_i$ and ideal I_1 on u_1,

$$\lambda = \mathrm{tcf}\left(\prod_{i\in u_1} \mu_i, \leqq_{I_1}\right) \quad \text{and} \quad [i \neq j \Rightarrow \mu_i \neq \mu_j].$$

(2) If $\lambda > 2^{|u|}$, we can have $\mu_i \in R^{|u|,2}$ (for $i \in u_1$).

(3) If I is σ-complete, we can have "I_1 σ-complete" and if I is normal ($u = \theta \in R^{\kappa,1}$) we can have "$I_1$ is normal ideal on θ".

(4) Suppose $f_\alpha \in \Pi_{i\in u} \lambda_i$ for $\alpha < \lambda$, I is an ideal on u and $[\alpha < \beta \Rightarrow f_\alpha <_I f_\beta]$, $\lambda > 2^{|u|}$, $\lambda > \lambda_i$ for $i \in u$, λ regular. *Then* we can find a set v, a function h from some u_1, $u - u_1 \in I$, onto v, an ideal J on v and regular cardinals μ_i for $i \in v$ s.t.:

(i) $J = \{h''(A) : A \in I\}$,

(ii) $\lambda = \mathrm{trf}(\Pi_{i\in v} \mu_i, \leqq_J)$,

(iii) $\mu_{h(i)} \leqq \lambda_i$ for $i \in u$,

(iv) if $i \in v$, $j \in v$, $i \neq j$ then $\mu_i \neq \mu_j$,

(v) $2^{|u|} < \mu_j$ for $j \in v$.

(5) If in (1) I is selective [i.e., for every equivalence relation E on u, for some i if $E \neq \varnothing \bmod I$ *or* for some $v \neq \varnothing \bmod I$, $|v \cap i/E| \leqq 1$ for $i \in u$ ($v \subseteq u$)], *then* we can have (i)′$v \subseteq u$, $J = \{w \cap v : w \in I\}$, $h \restriction v = \mathrm{id}_v$ [note: if $\kappa \leqq |u|$, then $\{v \subseteq u : |v| < |u|\}$ is selective, and if $u = \theta = \mathrm{cf}\,\theta > \aleph_0$ then $\mathscr{D}_\theta$ is selective].

(6) Suppose $f_\alpha \in \Pi_{i\in u} \lambda_i$ for $\alpha < \lambda$, J, I ideals on $u, J \subseteq I$, I is an $\aleph_1$-complete ideal on u, $[\alpha < \beta \Rightarrow f_\alpha \leqq_J f_\beta]$, λ is regular, $\bigwedge_{i\in u} \lambda_i < \lambda$, $|u| < \lambda$ and $\langle f_\alpha/I : \alpha < \lambda\rangle$ is not eventually constant and in $\mathscr{P}(u)/J$ there is no strictly increasing sequence of length λ (which remains so mod I) and I is $\aleph_1$-complete, *then*

(i) for some $v \subseteq u$, and pairwise distinct or equal regular cardinals μ_i ($i \in v$), $\mu_i \leqq \lambda_i$, and some filter $I(*)$ on v, $\mathrm{tcf}\,(\Pi_{i\in v} \mu_i \leqq_{I(*)}) = \lambda$, $I(*)$ $\aleph_1$-complete (and is σ-complete, normal, if J is),

(ii) $\mathrm{tcf}(\Pi\, \mu_i, \leqq_{I(*)}) = \lambda$ is witness by $\langle f_\alpha : \alpha < \lambda\rangle$ s.t. $\alpha < \beta \Rightarrow f_\alpha \leqq f_\beta \bmod J$.

PROOF. (1) by 5(2) and 9(1); and for "$\lambda_i \neq \lambda_j$" use 4(3). (We use 4(1) without saying.)

(2) Use 4(6).

(3) Use 9(2), 9(3), too.

(4) By 5(2), 4(3).

(5) Easy.

(6) By 4(9), wlog$[\alpha<\beta \Rightarrow f_\alpha <_I f_\beta]$ and then use 9(4) to have some g, $\bigwedge_{i\in u} g(i) \leqq \lambda_i$, and $\lambda \in \mathrm{psc}(\Pi_{i\in u} g(i), \leqq_I)$, then use 9(1) + 9(2) and lastly 4(3). (There is no problem to preserve normality too.) (When $J \neq I$, we have to look again at the proof of 9(1).)

10A. COROLLARY. (1) *Suppose* $\lambda > \mathrm{cf}\,\lambda > \aleph_0$, *$I$ is a normal ideal on* $\mathrm{cf}\,\lambda$ *such that* (*) *for no* $\langle u_\alpha : \alpha < \lambda \rangle : u_\alpha \subseteq \mathrm{cf}\,\lambda$, $[\alpha < \beta \Rightarrow |u_\beta - u_\alpha| < \mathrm{cf}\,\lambda]$, $[\alpha<\beta \Rightarrow u_\alpha \not\subseteq u_\beta \bmod I]$. Then

(i) *for some normal ideal J on* $\mathrm{cf}\,\lambda$, $I \subseteq J$, *and some regular* $\lambda_i < \lambda$ *for* $i < \mathrm{cf}\,\lambda$, $\langle \lambda_i : i < \mathrm{cf}\,\lambda\rangle$ *constant or strictly increasing*, $\lambda^+ = \mathrm{tcf}(\Pi_{i<\mathrm{cf}\,\lambda}\lambda_i, \leqq_J)$,

(ii) $\lambda^+ = \mathrm{tcf}(\Pi_{i<\mathrm{cf}\,\lambda}\lambda_i, \leqq_J)$ *is witnessed by some* $\langle f_\alpha : \alpha < \lambda^+\rangle$ *s.t. for* $\alpha < \beta$, $f_\alpha \leqq f_\beta \bmod \mathscr{D}^{\mathrm{cobd}}_{\mathrm{cf}\,\lambda}$,

(iii) *the λ_i's are all equal or* $\langle \lambda_i : i < \mathrm{cf}\,\lambda\rangle$ *is strictly increasing*.

(2) *We can replace "normal" by "σ-complete" if*: $\mathrm{cf}\,\lambda \geqq \sigma > \aleph_0$ *and in* (iii) *we assume* $I = \{v \subseteq \mathrm{cf}\,\lambda : |v| < \mathrm{cf}\,\lambda\}$.

PROOF. (1) Let $\langle \lambda_i : i < \mathrm{cf}\,\lambda\rangle$ be strictly increasing $<\lambda$ s.t. $\lambda = \Sigma_{i<\mathrm{cf}\,\lambda}\lambda_i$. By 9(5) we can find $\langle f_\alpha : \alpha < \lambda^+\rangle$ s.t.: $f_\alpha \in \Pi_{i<\mathrm{cf}\,\lambda}\lambda_i^+$ and $[\alpha<\beta \Rightarrow f_\alpha < f_\beta \bmod \mathscr{D}^{\mathrm{cobd}}_{\mathrm{cf}\,\lambda}]$. By 10(6) we can finish (with μ_i there as our desired λ_i's).

(2) Similarly,

11. COROLLARY. *If* $2^{\mathrm{cf}\,\lambda} < \lambda$ then *for some constant or strictly increasing sequence* $\langle \lambda_i : i < \mathrm{cf}\,\lambda\rangle$ *of regular cardinals* $<\lambda$ $\mathrm{tcf}(\Pi\lambda_i, \leqq_{\mathscr{D}^{\mathrm{cobd}}_{\mathrm{cf}\,\lambda}}) = \lambda^+$ (*witnessed by a* $\leqq_{\mathscr{D}^{\mathrm{cobd}}_{\mathrm{cf}\,\lambda}}$*-increasing sequence*).

PROOF. Choose λ_i, $\lambda = \Sigma\{\lambda_i : i < \mathrm{cf}\,\lambda\}$, $\langle \lambda_i : i < \mathrm{cf}\,\lambda\rangle$ as in 9(5) and then use 9(5), 10(4).

12. LEMMA. *If* $\sigma = \mathrm{cf}\,\mu > \aleph_0$, $(\forall\theta < \mu)[\theta^{<\sigma} \leqq \mu]$, $\mu < \lambda \leqq \mu^\sigma$, *$\lambda$ regular*, then *for some normal filter $\mathscr{D}$ on σ and* $\theta_i < \mu$ (*for* $i < \sigma$), $\mathrm{tcf}(\Pi_{i<\sigma}, \mathscr{D}) = \lambda$.

PROOF. By [Sh 4] §7.

13. LEMMA. (1) *In* 9(4) *if* ($u = \theta$ *a regular cardinal*) E, $\mathscr{D}$ *are nice*

(normal) filters on λ, θ *resp*. (see [Sh 4, 7.10]), then *for some nice* $\mathscr{D}_1$ *extending* $\mathscr{D}$,

$$\operatorname{tcf}\left(\prod_{i<\theta} \lambda_i, \leqq_{\mathscr{D}_1}\right) = \lambda.$$

(2) *If* λ *is singular, on* λ^+, $\operatorname{cf}\lambda$ *there are nice filters* E, $\mathscr{D}$ then *for some nice filter* $\mathscr{D}_1$, *extending* $\mathscr{D}$ *and* $\lambda_i < \lambda$ *(for* $i < \operatorname{cf}\lambda$)

$$\operatorname{tcf}\left(\prod_{i<\operatorname{cf}\lambda} \lambda_i/\mathscr{D}\right) = \lambda^+.$$

PROOF. (1) Let g_n be as in the proof of 9(4) and let for $\alpha < \lambda$

$$A_\alpha = \{i \leqq \theta : f_\alpha(i) < g_n(i)\}.$$

Clearly: $A_\alpha \neq \emptyset \mod \mathscr{D}$,

$$\alpha < \beta \Rightarrow A_\beta \subseteq A_\alpha \mod \mathscr{D}.$$

W.l.o.g. $(\forall i) g_n(i) > 0$.

Let $\mathscr{D} = \{A \subseteq \theta : \text{for every } \alpha, A \cup (\theta - A_\alpha) \in \mathscr{D}^+\}$. Clearly

$$\operatorname{tcf}\left(\prod_{i<\theta} \lambda_i, \leqq_{\mathscr{D}}\right) = \lambda.$$

Why is $\mathscr{D}$ nice? By observation 13A below.

(2) By 13(1) and 9(5).

13A. OBSERVATION. If $\lambda > \theta > \aleph_0$ are regular, E a nice filter on λ, and for $\alpha < \lambda$ $\mathscr{D}_\alpha$ is a nice filter on θ, then:

(i) $\mathscr{D} \stackrel{\text{def}}{=} \{A \subset \theta : \{\alpha < \lambda : A \in \mathscr{D}_\alpha\} \in E\}$ is nice,

(ii) if $f \in {}^\theta\mathrm{Ord}$, $h \in {}^\lambda\mathrm{Ord}$, $r\kappa(f, \mathscr{D}_\alpha) \leqq h(\alpha)$ for $\alpha < \lambda$, *then* $r\kappa(f, \mathscr{D}) \leqq r\kappa(h, E)$.

PROOF. See [Sh 4] Def. 5.1, 3.1

13B. REMARK. (1) Note that: if

$$(\exists \kappa)(\exists \mu)[\mu > 2^{2^{\operatorname{cf}\lambda}} \wedge \mu^\kappa > (\mu + \kappa)^+], \qquad \operatorname{cf}\lambda > \aleph_0$$

then there is a nice filter $\mathscr{D}$ on $\operatorname{cf}(\lambda)$ (see [Sh 4] 4.14, 4.15) and also on $I = \{a \subseteq \lambda : |a| < \theta\}$ if $\theta = \operatorname{cf}\theta > \aleph_0$, $\lambda^\theta = \lambda$.

(2) In fact, in (1) the club filter is nice (on λ and on $\{a \subseteq \lambda : |a| < \theta\}$).

(3) If $\lambda^\theta = \lambda$, $\aleph_0 < \sigma = \operatorname{cf}\sigma \leqq \theta$, and the club filter on $\{a \subseteq \lambda : |a| < \theta\}$ is nice, *then* every normal fine filter on $\{a \subseteq \theta : |a| < \sigma\}$ is nice.

§2. Good stationary sets and existence of lub's

14. LEMMA. (1) *If λ is singular, $S \subseteq \lambda^+$ is a stationary subset of λ^+ which is good (see below) and $\delta \in S \Rightarrow \mathrm{cf}(\delta) > \kappa \overset{\mathrm{def}}{=} \mathrm{cf}\,\lambda$, then for some $\lambda_i < \lambda$ (for $i < \kappa$), and a uniform ultrafilter $\mathscr{D}$ on κ, $\Pi_{i<\kappa}\lambda_i/\mathscr{D}$ has cofinality λ^+.*

(2) *Suppose λ_i is regular $< \lambda$ for $i < \kappa$, λ is regular, $\mathscr{D}$ a filter on κ, $\kappa < \lambda$, $f_\alpha \in \Pi_{i<\kappa}\lambda_i$, $[\alpha < \beta \Rightarrow f_\alpha <_{\mathscr{D}} f_\beta]$. Then one of the following occurs:*

(a) *$\langle f_\alpha : < \lambda \rangle$ has a $w <_{\mathscr{D}}$-lub f_λ (hence for some filter $\mathscr{D}_1$ extending $\mathscr{D}$ $[f <_{\mathscr{D}} f_\lambda \Rightarrow (\exists \alpha < \lambda) f <_{\mathscr{D}_1} f_\alpha]$).*

(b) *In $\mathscr{P}(\kappa)/\mathscr{D}$ there is a strictly increasing sequence $\langle a_\alpha/\mathscr{D} : \alpha < \lambda \rangle$ of length λ (in particular, $\mathscr{D}$ is not an ultrafilter).*

(c) *There is a club $B \subseteq \lambda$ and functions $g_\alpha \in \Pi_{i<\kappa}\lambda_i$ for $\alpha \in B$ such that*

(α) *$f_\alpha \leqq_{\mathscr{D}} g_\alpha \leqq_{\mathscr{D}} f_\beta$ for $\alpha < \beta$ in B (so you can get $f_\alpha <_{\mathscr{D}} g_\alpha <_{\mathscr{D}} f_\beta$),*

(β) *$\bigcup_{\alpha\in\beta} \mathrm{rang}\, g_\alpha$ has cardinality $\leqq \kappa$.*

(3) *If there is a good stationary set*

$$S \subseteq \{\delta < \lambda : \kappa < \mathrm{cf}\,\delta < \mathrm{Min}\{\lambda_i : i < \kappa\}\},$$

$\lambda, \lambda_i, \kappa, f_\alpha$ $(i < \kappa, \alpha < \lambda)$ as in (2), $\langle A_\alpha : \alpha < \lambda \rangle$ witnesses "S is good" and $[\alpha \in A_\beta \Rightarrow \bigwedge_i f_\alpha(i) < f_\beta(i)]$, then *in* (2) *above* (c) *and* (b) *do not occur.*

(4) *In* (3) *(using of) we can conclude $\lambda \rightarrow \{\lambda_i : i < \kappa\}$* ([Sh 1]), *so, e.g., if $\lambda = \aleph_{\omega+1}$ there is on λ a Jonsson algebra.*

(5) *In* (1), *if $\kappa < \lambda$, $\{\delta \in S : \mathrm{cf}(\delta) \geqq \kappa\}$ is stationary, then*

$$(\exists i < \mathrm{cf}\,\lambda)\forall j[i < j < \mathrm{cf}(\lambda) \Rightarrow \lambda_i > \kappa].$$

REMARK. On good sets see [Sh 8] and Appendix of [Sh 9]; they are defined in 15 below.

15. DEFINITION. For a regular λ, $S \subseteq \lambda$ is good *if* there is a sequence $\langle A_\alpha : \alpha < \lambda \rangle$, $A_\alpha \subseteq \alpha$ and for some closed unbounded $C \subseteq \lambda$ for every $\delta \in C \cap S$ there is an unbounded $A \subseteq \delta$ of order type $\mathrm{cf}\,\delta$ such that $(\forall \alpha < \delta)$ $[A \cap \alpha \in \{A_\beta : \beta < \alpha\}]$.

PROOF. (1) Let $\langle A_\alpha : \alpha < \lambda \rangle$ be as in Definition 15. W.l.o.g. for each limit α $\{A_\alpha \cap \gamma : \gamma \in A_\alpha\} \subseteq \{A_\beta : \beta < \alpha\}$. Note that w.l.o.g. $|A_\alpha| < \lambda$ for every α. Let $\langle \lambda_i : i < \kappa \rangle$ be an increasing continuous sequence of cardinals $< \lambda$ with $\lambda = \bigcup_{i<\kappa}\lambda_i$. We can define, by induction on $\alpha < \lambda^+$, a function f_α such that

(i) $f_\alpha \in \Pi_{i<\kappa}\lambda_i^+$;

(ii) for $\beta < \alpha$, for every large enough $i < \kappa$, $f_\beta(i) < f_\alpha(i)$;

(iii) if $\beta \in A_\alpha$, $i < \kappa$, $|A_\alpha| \leqq \lambda_i$ *then* $f_\beta(i) < f_\alpha(i)$.

Now $\langle f_\alpha : \alpha < \lambda^+ \rangle$ has a lub f_{λ^+} [as otherwise use part 3 of 14, i.e., for (c) find a closed unbounded $C \subseteq \lambda^+$ and $\langle g_\alpha : \alpha \in C \rangle$, $g_\alpha \in \Pi_{i<\kappa} A_i$, $|A_i| \leqq \kappa$, $f_\alpha <_{\mathscr{D}} g_\alpha <_{\mathscr{D}} f_\beta$ for $\alpha < \beta$ from C; choose $\delta \in S$ a limit point of C, and get easy contradiction by (iii) above]. By 4(1),(3), 9(1) we get the conclusion.

(2) We define, by induction on $\zeta < \kappa^+$, a function $g_\zeta \in \Pi_{i<\kappa}(\lambda_i + 1)$ such that $[\xi < \zeta \Rightarrow g_\zeta \leqq_{\mathscr{D}} g_\xi]$, $[\zeta = \xi + 1 \Rightarrow g_\xi \neq_{\mathscr{D}} g_\zeta]$ and $[\alpha < \lambda \Rightarrow f_\alpha \leqq_{\mathscr{D}} g_\zeta]$. We let g_0 be defined by $g_0(i) = \lambda_i$. If $\langle g_\xi : \xi \leqq \zeta \rangle$ are defined, and there is g, $g \in \Pi_{i<\kappa}(\lambda_i + 1)$, $f_\alpha \leqq_{\mathscr{D}} g$ for $\alpha < \lambda$, $g \neq_{\mathscr{D}} g_\zeta$ and $g \leqq_{\mathscr{D}} g_\zeta$, then choose such a function g as $g_{\zeta+1}$; if we cannot, we have gotten "$\langle f_i : i < \lambda \rangle$ has a $w <_{\mathscr{D}}$-lub", hence (a) holds. If ζ is limit ordinal, $\zeta \leqq \kappa^+$, let $A_i^\zeta = \{g_\xi(i) : \xi < \zeta\} \cup \{\lambda_i\}$, so A_ζ^i is a set of $\leqq |\zeta| + 1$ ordinals. For $\alpha < \lambda$, let $f_\alpha^\zeta \in \Pi_{i<\kappa}(\lambda_i + 1)$ be defined by

$$f_\alpha^\zeta(i) = \mathrm{Min}\{\gamma \in A_i^\zeta : f_\alpha(i) \leqq \gamma\}$$

(well defined as $f_\alpha(i) < \lambda_i \in A_i^\zeta$). Clearly $\alpha < \lambda \Rightarrow f_\alpha \leqq_{\mathscr{D}} f_\alpha^\zeta$ and also $[\alpha < \beta < \lambda \Rightarrow f_\alpha^\zeta \leqq_{\mathscr{D}} f_\beta^\zeta]$.

Case I: $\zeta < \kappa^+$ and $(\forall \alpha < \lambda)(\exists \beta < \lambda)[\alpha < \beta \wedge f_\alpha^\zeta <_{\mathscr{D}} f_\beta^\zeta]$ then conclusion (c) holds.[5]

Case II: $\zeta < \kappa^+$ and for some $\alpha = \alpha_\zeta^0$,

$$(\forall \beta < \lambda)[f_\alpha^\zeta \not<_{\mathscr{D}} f_\beta^\zeta] \text{ but } (\forall \alpha < \lambda)(\exists \beta < \lambda)[\alpha < \beta \wedge f_\beta^\zeta \not<_{\mathscr{D}} f_\alpha^\zeta].$$

Then some closed unbounded subsequence of $\langle a_\beta : \beta < \lambda \rangle$ exemplifies (b) where

$$a_\beta = \{i : f_\beta^\zeta(\lambda_i) > f_{\alpha_\zeta^0}^\zeta(\lambda_i)\}.$$

Case III: $\zeta < \kappa^+$ and both previous cases fail, then for some $\alpha = \alpha_\zeta$ $(\forall \beta < \lambda)[\alpha \leqq \beta \Rightarrow f_\alpha^\zeta =_{\mathscr{D}} f_\beta^\zeta]$.

We let g_ζ be $f_{\alpha_\zeta}^\zeta$.

Case IV: Suppose $\zeta = \kappa^+$ [so $\langle g_\xi : \xi < \kappa^+ \rangle$ are defined and for $\alpha < \lambda$ and limit $\xi < \kappa^+$, α_ξ and f_α^ξ are defined].

Note that $\alpha^* \overset{\text{def}}{=} \bigcup\{\alpha_\xi : \xi < \kappa^+\}$ is $< \lambda$.

Note that for each $\alpha < \lambda$, $i < \kappa$, $f_\alpha^\zeta(i) \in A_i^\zeta = \bigcup_{\xi<\kappa^+} A_i^\xi$, so for some $\xi = \xi(\alpha, i) < \kappa^+$, $f_\alpha^\zeta(i) \in A_i^{\xi(\alpha,i)}$. Let $\xi(\alpha) = \sup_{i<\kappa} \xi(\alpha, i)$, it is $< \kappa^+$, and as $\langle A_i^\xi : \xi < \kappa^+ \rangle$ is increasing, $f_\alpha^\zeta(i) \in A_i^{\xi(\alpha)}$. Clearly then, e.g., $f_\alpha^{\xi(\alpha)+\omega} = f_\alpha^{\xi(\alpha)+\omega+\omega}$

[5] Note that $f_\alpha^\zeta <_{\mathscr{D}} f_\beta^\zeta$ implies $f_\alpha^\zeta <_{\mathscr{D}} f_\beta$.

hence using $\alpha \geqq \alpha^*$, $g_{\xi(\alpha)+\omega} =_{\mathscr{D}} g_{\xi(\alpha)+\omega+\omega}$ hence $g_{\xi(\alpha)+\omega+1} \leqq_{\mathscr{D}} g_{\xi(\alpha)+\omega}$, contradiction to the choice of $g_{\xi(\alpha)+\omega+1}$.

(3), (4), (5) Left to the reader.

§3. More on cofinalities

16. Lemma. *Let* $a \in I^{\kappa,2}$, $\lambda \in R^{\kappa,2}$.

(1) *For some set* $b \subseteq a$, $J^{\kappa,a}_{<\lambda^+} = (J^{\kappa,a}_{<\lambda}) + b$ [*we choose such a set b and call it* $b^{\kappa,a}_\lambda$, *and w.l.o.g. if* $a \in J^{\kappa,a}_{<\lambda^+}$ *then* $b = a$ *and if* $J^{\kappa,a}_{<\lambda^+} = J^{\kappa,a}_{<\lambda}$ *then* $b^{\kappa,a}_\lambda = \varnothing$].

(2) $J^{\kappa,a}_{<\lambda}$ *is the ideal (of subsets of a) generated by* $\{b^{\kappa,a}_\mu : \mu \in R^{\kappa,2}, \mu < \lambda\}$.

(3) pcf(a) *has cardinality* $\leqq 2^\kappa$ *and has a maximal element, which is equal to* cf$(\Pi a, \leqq)$.

(4) *For* $b = b^{\kappa,a}_\lambda \neq \varnothing$ *there are functions* $f^{\kappa,a}_{\lambda,\alpha}$ $(\alpha < \lambda)$ *which witness* "tcf$(\Pi b, \leqq_{J^{\kappa,a}_\lambda}) = \lambda$" (*though the choice of* $f^{\kappa,a}_{\lambda,\alpha}$ *is not unique, we choose such functions*). *W.l.o.g. if* $\langle f^{a}_{\lambda,\alpha} \restriction b^{\kappa,a}_\lambda : \alpha < \delta < \lambda \rangle$ *has a least upper bound* mod $J^{\kappa,a}_{<\lambda}$, *then* $(f^{\kappa,a}_{\lambda,\delta} \restriction b^{\kappa,a}_\lambda)/J^{\kappa,a}_{<\lambda}$ *is such a least upper bound.*

(5) *For every* $f \in \Pi a$ there are $n > 0$, $\lambda_0 > \lambda_1 > \cdots > \lambda_n$ *from* pcf(a), $\alpha_0 < \lambda_0$, $\alpha_1 < \lambda_1, \ldots, \alpha_n < \lambda_n$, *and* $b_l \subseteq a$ *for* $l \leqq n$ *such that* $\langle b_l : l \leqq n \rangle$ *is a partition of a, and*

$$b_0 \equiv b^{\kappa,a}_{\lambda_0} \bmod J^{\kappa,a}_{<\lambda_0}, \quad b_l \equiv b^{\kappa,a}_{\lambda_l} \bmod J^{\kappa,a}_{<\lambda_l}, \quad f \restriction b_l \leqq f^{\kappa,a}_{\lambda_l,\alpha_l} \restriction b_l$$

(*note*: $a - b_0 \in J^{\kappa,a}_{<\lambda_0}$, $a - b_0 \cup \cdots \cup b_l \in J^{\kappa,a}_{<\lambda_l}$, $a - b_0 \cup \cdots \cup b_n = \varnothing$).

(6) *Let* $\mathscr{D}$ *be an ultrafilter on a*: cf$(\Pi a, <_{\mathscr{D}}) = \lambda$ *iff* $\lambda \in R^{\kappa,2}$ *and* $b^{\kappa,a}_\lambda \in \mathscr{D}$, $J^{\kappa,a}_{<\lambda} \cap \mathscr{D} = \varnothing$.

Proof. By [Sh 5] Ch. XIII, §5, §6.

17. Claim. (1) $a \subseteq$ pcf(a), [$a \subseteq b \Rightarrow$ pcf$(a) \subseteq$ pcf(b)];

(2) $\mu \in$ pcf$(a) \Rightarrow \mu \in$ pcf$(a \cap \mu^+)$;

(3) for $\mu > 2^{|a|}$, $\mu \in$ pcf(pcf$(a)) \Leftrightarrow \mu \in$ pcf(a);

(4) pcf$(a - \{\theta\}) \supseteq$ pcf$(a) - \{\theta\}$;

(5) if $\lambda \in$ pcf(a), $\lambda > 2^\kappa$, $a \in I^{\kappa,2}$ then $\lambda \in$ pcf$(a - (2^\kappa)^+)$ (we can use any $\mu = \mu^\kappa$ instead of 2^κ) and $\lambda = \max$ pcf$(b^{\kappa,a}_\lambda)$;

(6) if $\theta \in$ pcf(a), $a \in I^{\kappa,2}$ and $(\forall \alpha < \theta)[|\alpha|^{<\sigma} < \theta]$ then $J^{\kappa,b}_{<\theta}$ is σ-complete for $b = b^{\kappa,a}_\theta$.

Proof. (1) Easy (see Definition 6(1)).

(2) If $\mathscr{D}$ is an ultrafilter on a, and $\{\theta \in a : \theta > \mu\} \in \mathscr{D}$, then easily tcf$(\Pi a, <_{\mathscr{D}}) > \mu$. Now as $\mu \in$ pcf(a), for some ultrafilter $\mathscr{D}$ on a, tcf$(\Pi a, <_{\mathscr{D}}) = \mu$ hence $(a - \mu^+) \notin \mathscr{D}$ hence $a \cap \mu^+ \in \mathscr{D}$, hence $\mathscr{D}_1 =$

$\mathscr{D} \upharpoonright (a \cap \mu^+)$ is an ultrafilter on $a \cap \mu^+$ and $\mathrm{cf}(\Pi(a \cap \mu^+), <_{\mathscr{D}_1}) = \mu$, so $\mu \in \mathrm{pcf}(a \cap \mu^+)$.

(3) By 17(1), $\mu \in \mathrm{pcf}(a) \Rightarrow \mu \in \mathrm{pcf}(\mathrm{pcf}(a))$. So assume $\mu \in \mathrm{pcf}(\mathrm{pcf}(a))$. Let $\mathrm{pcf}(a) = \{\mu_i : i < i(*)\}$, so $i(*) < (2^{|a|})^+$, for each i there is an ultrafilter $\mathscr{D}_i$ on a such that $\mu_i = \mathrm{tcf}(\Pi a, <_{\mathscr{D}_i})$. So $\mu = \mathrm{tcf}(\Pi \mu_i, \leqq_{\mathscr{D}})$, $\mathscr{D}$ an ultrafilter on $i(*)$. By the theorem of ultraproducts of ultraproducts

$$\mu = \mathrm{tcf}(\Pi\{\theta_j : \langle j, i\rangle \in j(*) \times i(*)\}, \leqq_{\mathscr{D}^*})$$

where

$$\mathscr{D}^* = \{A \subseteq j(*) \times i(*) : \{j < j(*) : \{i : i < i(*) \text{ and } \langle j, i\rangle \in A\} \in \mathscr{D}_j\} \in \mathscr{D}\}.$$

By 4(3)(c) we finish.

(4), (5) Left to the reader.

(6) Suppose $b_i \in J^{\kappa,b}_{<\theta}$ for $i < i(*)$, $i(*) < \sigma$, $\bigcup_i b_i \notin J^{\kappa,b}_{<\theta}$; w.l.o.g. $b = \bigcup_i b_i$ and the b_i's are pairwise disjoint. Let $F_i \subseteq \Pi(b_i)$ be a cover of cardinality $< \lambda$ (exists as we know $\mathrm{cf}(\Pi b_i) < \theta$ as $b_i \in I^{\kappa,b}_{<\theta}$ by 16(3)). Now

$$F = \left\{ \bigcup_{i < i(*)} f_i : f_i \in F_i \text{ for } i < i(*) \right\}$$

has cardinality $\leqq \Pi_{i < i(*)} |F_i| < \theta$ by our assumptions and F is a cover of Πb, contradiction.

17A. Lemma. *Suppose $\sigma = \mathrm{cf}\,\mu > \aleph_0$, $(\forall \theta < \mu)\ [\theta^\sigma \leqq \mu]$, $\mu < \lambda = \mathrm{cf}\,\lambda \leqq \mu^\sigma$, $2^\sigma < \mu$.*

(1) *If $\lambda = \mathrm{cf}\,\lambda < \mu^{+\sigma^+}$ then we can have $\langle \theta_i : i < \sigma\rangle$ strictly increasing and*

$$\theta_i < \mu, \qquad \mathrm{tcf}(\Pi \theta_i, <_{J^{bd}_\sigma}) = \lambda.$$

(2) *If $[a \in R^{\sigma,2}, a \subseteq \mu \Rightarrow |\mathrm{pcf}(a)| \leqq \sigma]$ then the conclusion of (2) holds.*

(3) *We can allow $\sigma = \mathrm{cf}\,\mu = \aleph_0$ if we add $\lambda \in \mathrm{pcf}(a)$, $a \in \mu$, $a \in R^{\sigma,2}$.*

Proof. (1) By the proof of (2).

(2) As in 12(1), we can find $\langle \lambda_i : i < \sigma\rangle$, $\mathscr{D}$ s.t.: $\langle \lambda_i : i < \sigma\rangle$ is strictly increasing, $\mathscr{D}$ a normal filter on σ, $\lambda_i < \mu$ and $\mathrm{tcf}(\Pi_{i<\sigma} \lambda_i, \leqq_{\mathscr{D}}) = \lambda$. Let $a = \{\lambda_i : i < \sigma\}$, clearly $J^{a,\sigma}_{<\mu^+} = J^{bd}_a$, let $\mathrm{pcf}(a) - \mu = \{\mu_\alpha : \alpha \leqq \alpha(*)\}$ (μ_α increasing), so for some γ, $\lambda = \mu_\gamma$, and necessarily $\bigwedge_{a<\gamma}(a - b^{\kappa,a}_{\mu_\alpha}) \in \mathscr{D}$. As $\gamma < \sigma^+$ (by an assumption) and $\mathscr{D}$ is normal, there is $a^* \in \mathscr{D}$ such that $a^* \subseteq b^{\kappa,c}_{\mu_\gamma}$ and $[\alpha < \gamma \Rightarrow |a^* \cap b^{\kappa,a}_{\mu_\alpha}| < \sigma]$.

17B. LEMMA. *If* $\lambda < \mu$ *belongs to* pcf(a), $2^\kappa < \lambda$, $a \in R^{\kappa,2}$ *and* pcf$(a) \cap [\lambda, \mu) = $ pcf(pcf$(a) \cap [\lambda, \mu)$) then *for some* b

$$J^{a,\kappa}_{<\mu} = (J^{a,\kappa}_{<}) + b.$$

PROOF. Easy.

REMARK. Below, ac stands for "accessible", ia stands for "inaccessible".

18. DEFINITION. (1) $R^{\kappa,2}_{ac,\sigma} = \{\theta \in R^{\kappa,2}$: for some $a \in I^{\kappa,2}$, $\theta \in \mathrm{pcf}_\sigma(a)$ and $\theta > \sup(a)\}$.

(2) $R^{\kappa,2}_{ia,\sigma} = R^{\kappa,2} - R^{\kappa,2}_{ac,\sigma}$.

(3) If $\sigma = \aleph_0$ we omit it.

(4) Instead of σ we can use Γ (see 6(1)(b)).

19. DEFINITION. (1) $R^{\kappa,l}_{ac,\Gamma} = \{\theta \in R^{\kappa,l}$: for some $\bar\lambda \in \mathcal{I}^{k,l}$, $\theta \in \mathrm{pcf}_\Gamma(a)$ and $(\forall i \in \mathrm{dom}\,\bar\lambda)\lambda_i < \theta\}$

(2) $R^{\kappa,l}_{ia,\Gamma} = R^{\kappa,l} - R^{\kappa,l}_{ac,\Gamma}$.

(3) If Γ is being σ-complete we write σ.

(4) If $\sigma = \aleph_0$ we omit it.

20. REMARK. A disturbing point of $R^{\kappa,1}$ is that 17(5) is not known.

21. FACT. Definitions 18 and 19 are compatible.

22. LEMMA. *If* $\theta \in R^{\kappa,2}_{ac}$ then *for some* $a \in I^{\kappa,2}$: $\theta \in \mathrm{pcf}(a)$ *and* $a \subseteq R^{\kappa,2}_{ia}$.

PROOF. We define by induction on $n < \omega$ the following T_n, and a_η, $\mathcal{D}_\eta$ (for $\eta \in T_n$) such that:

(i) $T_0 = \{\langle\theta\rangle\}$.

(ii) T_n is a set of decreasing sequences of cardinals from $R^{\kappa,2}$, each sequence has length $n+1$ and $|T_n| \leqq \kappa$.

(iii) $\eta \in T_{n+1}$ implies $\eta \restriction (n+1) \in T_n$.

(iv) For $\eta \in T_n$, $a_\eta = \{\mu : \eta^\frown\langle\mu\rangle$ belongs to $T_{n+1}\}$.

(v) If $\eta \in T_n$, $a_\eta \neq \emptyset$ *then* $\mathcal{D}_\eta$ is an ultrafilter on a_η, $\mathrm{cf}(\Pi a_\eta, \leqq_{\mathcal{D}_\eta}) = \eta(n)$.

(vi) For $\eta \in T_n$, $a_\eta = \emptyset$ then $\eta(n) \in R^{\kappa,l}_{ia}$.

There is no problem to carry the definition.

Also letting $a^* = \{\eta(n) : n < \omega, \eta \in T_n, a_\eta = \emptyset\}$ easily $|a^*| \leqq \kappa$, $a^* \in I^{\kappa,2}$ and $\theta \in \mathrm{pcf}(a^*)$.

23. REMARK. The same proof gives 24 below (we can even replace $R^{\kappa,2}$ by any $R \subseteq R^0$ [interestingly Γ may be "σ-complete", "nice"; note they are closed under the sum operation below].

24. LEMMA. *Let Γ be a property of filters, $\theta \in R^{\kappa,l}_{ac,\Gamma}$. Then there is $\bar\lambda = \langle \lambda_i : i \in u \rangle$ and filter $\mathscr{D}$ on u such that*

(a) $\theta = \mathrm{tcf}(\Pi_{i \in u} \lambda_i, <_{\mathscr{D}})$,

(b) $\lambda_i < \lambda$, $\lambda_i \in R^{\kappa,l}_{ia,\Gamma}$,

(c) *there is a set Y of finite sequences of cardinality $\leqq \kappa$, closed under initial segments, $\langle\ \rangle \in Y$, which is well founded, $u = \{\eta \in Y$: for no $\upsilon \in Y$, η is a proper initial segment of $\upsilon\}$, $\langle\ \rangle \notin u$, and there are filters $\mathscr{D}_\eta$ on $Y_\eta = \{\eta^\wedge\langle x\rangle : \eta^\wedge\langle x\rangle \in Y\}$ for $\eta \in Y - u$ such that $\vDash \Gamma(\mathscr{D}_\eta)$ and for $X \subseteq u : X \in \mathscr{D}$* iff *for some $X' \subseteq Y$, $X' \cap u = X$, $\langle\ \rangle \in X'$ and*

$$[\eta \in X' \Rightarrow \{\eta^\wedge\langle x\rangle : \eta^\wedge\langle x\rangle \in X'\} \in \mathscr{D}_\eta].$$

§4. Representation by nice products

25. MAIN LEMMA. *Suppose $a \in I^{\kappa,2}$, $a \subseteq R^{\kappa,2}_{ia}$, $\lambda = \max \mathrm{pcf}(a)$, $\lambda > \sup(a)$, and for $\mu < \sup(a)$, $\sup \mathrm{pcf}(a \cap \mu) < \lambda$. Then there are $f_\alpha \in \Pi a$ for $\alpha < \lambda$ which witness $\mathrm{tcf}(\Pi a, \leqq_{J^{\kappa,a,2}_{<\lambda}}) = \lambda$ and*

(∗) *for $\mu \in a$, $\{f_\alpha \restriction (a \cap \mu) : \alpha < \lambda\}$ has cardinality $< \mu$*

(and f_δ is $<_{J^{\kappa,a}_{<\lambda}}$-lub of $\{f_\alpha : \alpha < \delta\}$ when $\mathrm{cf}\,\delta > 2^\kappa$).

REMARK. Note that a has no last element.

PROOF. Let $a = \{\mu_i : i < \delta(*)\}$, μ_i increasing with i, $a(\zeta) = a_\zeta = \{\mu_i : i < \zeta\}$ for $\zeta < \delta(*)$ ($\delta(*)$ is a limit ordinal as $\sup \mathrm{pcf}(a \cap \mu) < \lambda$ for $\mu < \sup(a)$). Let F_ζ be the set of functions of the form $f_0 \restriction b_0 \cup \cdots \cup f_n \restriction b_n$ where $\langle b_l : l \leqq n\rangle$ is a partition of a_ζ and $b_l \equiv b^{a,\kappa}_{\theta_l} \bmod J^{\kappa,a}_{<\theta_l}$ and

$$f_l \in \{f^{\kappa,a(\xi)}_{\theta_l,\alpha} : \xi \leqq \zeta, \theta_l \in \mathrm{pcf}(a_\xi), \alpha < \theta_l\}.$$

Clearly $|F_\zeta| \leqq 2^\kappa + \Sigma_{\xi \leqq \zeta} \sup \mathrm{pcf}(a_\xi) = 2^\kappa + \max \mathrm{pcf}(a_\zeta)$ which is $< \mu_\zeta$ as $[a \subseteq R^{\kappa,2} \Rightarrow 2^\kappa < \min(a)]$ and as $a \subseteq R^{\kappa,2}_{ia}$ by

26. FACT. If $\lambda \in R^{\kappa,2}_{ia}$, $a \subseteq R^{\kappa,2}$ (or even $a \subseteq R^{\kappa,0}$), $(\forall \mu \in a)[\mu < \lambda]$, then $\lambda > \max \mathrm{pcf}(a)$.

[PROOF. If $\lambda \leqq \max \mathrm{pcf}(a)$, $a \subseteq R^{\kappa,0}$ then there is an ultrafilter $\mathscr{D}$ on a, and in Πa, a $<_{\mathscr{D}}$-increasing sequence $\langle f_\alpha : \alpha < \max \mathrm{pcf}(a)\rangle$, so by 4(1), 5(2) for some $b \subseteq R^{\kappa,0}$, $|b| \leqq \kappa$, $\sup(b) \leqq \sup(a)$ ($< \lambda$) and $\lambda \in \mathrm{pcf}(b)$. By 4(6) w.l.o.g. $b \subseteq R^{\kappa,2}$, so we get a contradiction to $\lambda \in R^{\kappa,2}_{ia}$.]

Now it is enough to show

(*) for every $\alpha<\lambda$ for some $f^*_\alpha \in \Pi_{\zeta<\delta(*)}\mu_\zeta$:

$$\text{(i)}\ f^{\kappa,a}_{\lambda,\alpha} \leqq_{J^{\kappa,a}_{<\lambda}} f^*_\alpha \in \prod_{\zeta<\delta(*)} \mu_\zeta:$$

$$\text{(ii) for every } \zeta<\delta(*), f^*_\alpha \upharpoonright a_\zeta \in F_\zeta.$$

[Why is (*) enough? We then can choose by induction on α, β_α, f^*_α such that

(a) $\beta_\alpha<\lambda$ is such that $(\forall\gamma<\alpha) f^*_\gamma <_{J^{\kappa,a}_{<\lambda}} f^{\kappa,a}_{\lambda,\beta_\alpha}$,

(b) $f^*_\alpha \in \Pi_{\zeta<\delta(*)}\mu_\zeta$ is such that $f^{\kappa,a}_{\lambda,\beta_\alpha} \leqq_{J^{\kappa,a}_{<\lambda}} f^*_\alpha$ and f^*_α satisfies (ii),

the first step by the choice of $\langle f^{\kappa,a}_{\lambda,\alpha}:\alpha<\lambda\rangle$, the second by (*). Clearly $\langle f^*_\alpha:\alpha<\lambda\rangle$ satisfies the conclusion. More accurately, when α is limit of cofinality $>2^\kappa$ we want also that f^*_δ is $<_{J^{\kappa,a}_{<\lambda}}$-lub of $\{f^*_\beta:\beta<\delta\}$, but the proof of this is by 27 (i.e., 27 say this holds for a clus of δ's and now rename).]

Proof of (*). W.l.o.g. $[\theta\in a \Rightarrow \theta>(2^\kappa)^+]$. Let χ be a regular cardinal, large enough. We choose by inductions on $i<(2^\kappa)^+$ a model N_i such that:

(a) N_i is an elementary submodel of $(H(\chi),\in)$,

(b) $\|N_i\| \leqq 2^\kappa$ but $\{j: j<2^\kappa+i\}\subseteq N_i$,

(c) $\bigcup_{j<i} N_j \subseteq N_i$,

(d) the following belongs to N_{i+1}:

  (i) $\langle N_j: j\leqq i\rangle$,

  (ii) pcf(a) (hence pcf$(a)\subseteq N_{i+1}$),

  (iii) $\langle\langle f^{\kappa,b}_{\theta,\alpha}:\alpha<\theta\rangle:\theta\in \mathrm{pcf}(b), b\in R^{\kappa,2}, b\subseteq\lambda\rangle$,

  (iv) α (see (*)),

(e) for i limit, $N_i=\bigcup_{j<i}N_j$.

We now define $g_\gamma\in\Pi a$ for $\gamma\leqq(2^\kappa)^+$ by

$$g_\gamma(\mu_\zeta)=\sup\left(\mu_\zeta\cap\left(\bigcup_{i<\gamma} N_i\right)\right),$$

and for each $\theta\in R^{\kappa,2}\cap N_\gamma$ let $\delta^\gamma_\theta=\sup(\theta\cap(\bigcup_{i<\gamma}N_i))$. As we have assumed $[\theta\in a\Rightarrow\theta>(2^\kappa)^+]$ clearly $g_\gamma(\mu_\zeta)<\mu_\zeta$ for $\zeta<\delta(*)$ and $\delta^\gamma_\theta<\theta$ for $\theta\in R^{\kappa,2}\cap N_\gamma$.

From Fact 27 below we can prove (*), as $f^{\kappa,a}_{\lambda,\alpha}\in N_0$ and, e.g., $g_{(2^\kappa)^+}$ is an f^*_α as required. (Really for a closed unbounded set of $\gamma<(2^\kappa)^+$, cf $\gamma>\kappa$ implies g_γ is as required.)

27. Fact. Suppose $\langle N_i: i\leqq\delta\rangle$ is an increasing continuous sequence of elementary submodels of $(H(\chi),\in)$, $N_i\in N_{i+1}$. Define a function g_i: dom $g_i = R^0\cap N_i - \|N_i\|^+$, $g_i(\theta)=\sup(\theta\cap N_i)$; note that necessarily $g(\theta)<\theta$ as θ

is regular $> \| N_i \|$. Let $g = g_\delta$. Suppose further that $a \subseteq R^0 \cap N_\delta - |a|^+$, $a \in N_\delta$, $|a| \leqq \kappa$,

$$\langle \langle f^{\kappa,a}_{\lambda,\alpha} : \alpha < \lambda \rangle : \lambda \in \mathrm{pcf}(a) \rangle \in N_\delta, \qquad \langle b^{\kappa,a}_\theta : \theta \in \mathrm{pcf}(a) \rangle \in N_\delta.$$

Then:

(1) If $\theta \in \mathrm{pcf}(a)$, $\theta \in N_\delta$, $\mathrm{cf}(\delta) > 2^\kappa$ then

$$\bigoplus_\theta \; g \upharpoonright b^{\kappa,a}_\theta = f^{\kappa,a}_{\theta,g(\theta)} \upharpoonright b^{\kappa,a}_\theta \bmod I^{\kappa,2}_{<\theta}.$$

(2) If $\theta \in \mathrm{pcf}(a)$, $\theta \in N_\delta$, $I^{\kappa,2}_{<\lambda} \upharpoonright b^{\kappa,a}_\lambda$ is σ-complete and $\sigma > \mathrm{cf}(\delta)$ *then* $\bigoplus_\theta$.

(3) If $I^{\kappa,2}_{<\lambda} \upharpoonright b^{\kappa,a}_\lambda$ is generated by $\leqq \sigma$ sets (as an ideal) and $\mathrm{cf}(\delta) > \sigma$ *then* $\bigoplus_\theta$.

(4) If $\mathrm{cf}(\delta) > 2^\kappa$ (or for each $\theta \in N_\delta$, the assumptions of (2) or (3) hold) *then* for some $n < \omega$, $\theta_0 > \theta_1 > \cdots > \theta_n$ from $\mathrm{pcf}(a)$, $\theta_0 = \max \mathrm{pcf}(a)$ and for some partition $\langle b_l : l \leqq n \rangle$ of a,

$$g \upharpoonright a = \bigcup_{l=0}^{n} (f^{\kappa,a}_{\theta_l, g_\delta(\theta_l)} \upharpoonright b^{\kappa,a}_{\theta_l}).$$

(5) If the assumptions of (1) or (2) or (3) hold, $\mathrm{cf}(\delta) > \aleph_0$, *then* for a closed unbounded set of $\delta_1 < \delta$, $\mathrm{cf}(\delta_1) > \kappa$ implies

$$g_{\delta_1} \upharpoonright b^{\kappa,a}_\theta = f^{\kappa,a}_{\theta, g_{\delta_1}(\theta)} \upharpoonright b^{\kappa,a}_\theta \bmod I^{\kappa,2}_{<\theta}.$$

PROOF. (1) W.l.o.g. $a \cup \{a, \langle \langle f^{\kappa,a}_{\lambda,\alpha} : \alpha < \lambda \rangle : \lambda \in \mathrm{pcf}(a) \rangle, \theta\} \subseteq N_0$ so $a \subseteq \mathrm{dom}(g_i)$ for each i. Also as $N_i \in N_{i+1}$, $g_i \upharpoonright a \in N_{i+1}$. So for $i < j < \delta$, $g_i \upharpoonright a \in N_j$, hence $g_i(\mu) \in N_j$ for $\mu \in a$ hence $g_i \upharpoonright a < g_j \upharpoonright a$. Also $f^{\kappa,a}_{\theta,g_i(\theta)} \in N_j$ hence $f^{\kappa,a}_{\theta,g_i(\theta)}(\mu) \in N_j$ for $\mu \in a$ hence $f^{\kappa,a}_{\theta,g_i(\theta)} < g_j \upharpoonright a$. Lastly, as $g_i \upharpoonright a \in N_j$ and for some $\alpha < \theta$,

$$g_i \upharpoonright b^{\kappa,a}_\theta < f^{\kappa,a}_{\theta,\alpha} \upharpoonright b^{\kappa,a}_\theta \bmod I^{\kappa,2}_{<\theta}$$

this holds for some $\alpha \in N_j$ too, hence it holds for $\alpha = g_j(\theta)$. Now for each $i < \delta$, there is $c_i \subseteq b^{\kappa,a}_\theta$, $c_i \equiv b^{\kappa,a}_\theta \bmod I^{\kappa,2}_{<\theta}$, and

$$g_i \upharpoonright c_i \leqq f^{\kappa,a}_{\theta, g_{i+1}(\theta)} \upharpoonright c_i < g_{i+2} \upharpoonright c_i.$$

So for some c, $A = \{i < \delta : c_i = c\}$ is unbounded, and now by 4(7) $\{f^{\kappa,a}_{\theta,\alpha} \upharpoonright b^{\kappa,a}_\theta : \alpha < g_\delta(\theta)\}$ has a least upper bound $\mathrm{mod}(I^{\kappa,2}_{<\theta} \upharpoonright b^{\kappa,a}_\theta) : g \upharpoonright b^{\kappa,a}_\theta$ so we can finish.

(2), (3) Similar.

(4) Like the proof of 16(5).

(5) Left to the reader.

28. Main Lemma. *If* $\lambda = \max \mathrm{pcf}(a) > \sup(a)$, $a \in R^{\kappa,2}$, $\langle a_i : i < \sigma \rangle$ *is a partition of* a, *and* $(\forall i < \sigma) a_i \in J^{\kappa,a}_{<\lambda}$, then *for some* $n_i < \omega$ *and* $\theta_{i,0}, \ldots, \theta_{i,n(i)-1}$ *from* $\mathrm{pcf}(a_i)$ (*for* $i < \sigma$),

$$\lambda \in \mathrm{pcf}(\{\theta_{i,l} : i < \sigma, l < n(i)\}).$$

Proof. Let $\lambda(0) = \sup(\lambda \cap \mathrm{pcf}(a))$. As $|\mathrm{pcf}(a)| \leqq 2^\kappa < \lambda$ (as $\lambda > \sup(a)$, $a \in I^{\kappa,2}$) clearly $\lambda(0)$ is $< \lambda$. (We could obtain $\lambda(0) \notin \mathrm{pcf}(a)$ but have no need.) Let χ be a regular cardinal large enough and M an elementary submodel of $(H(\chi), \in)$ such that:

$$a \in M, \quad \{i : i \leqq \lambda(0)\} \subseteq M, \quad \text{and} \quad \| M \| = \lambda(0),$$
$$\langle \langle f^{\kappa,b}_{\theta,\alpha} : \alpha < \theta \rangle : b \subseteq a, \theta \in \mathrm{pcf}(b) \rangle \in M.$$

We now assume that the conclusion fails, and eventually we shall get a contradiction.

It is enough to show that for every $f \in \Pi a$ for some $g \in (\Pi a) \cap M$, $f \leqq g \bmod J^{\kappa,a}_{<\lambda}$. So let $f \in \Pi a$ be fixed.

We let $\gamma(*) = (2^\kappa)^+$, and $\langle N_i : i \leqq \gamma(*) \rangle$ be an increasing continuous sequence of elementary submodels of $(H(\chi), \in)$, such that:

(i) $\lambda, a, M, \langle \langle f^{\kappa,b}_{\mu,\alpha} : \alpha < \mu \rangle : \mu \in \mathrm{pcf}(b), b \subseteq a \rangle$ belong to N_0,

(ii) $\langle N_j : j \leqq i \rangle \in N_{i+1}$,

(iii) $\| N_i \| = 2^\kappa + |i|$,

(iv) $\{\alpha : \alpha < 2^\kappa + i\} \subseteq N_i$.

Clearly $a \subseteq \mathrm{pcf}(a) \subseteq N_0$ and every subset of $\mathrm{pcf}(a)$ of cardinality $\leqq \kappa$ belongs to N_0. Let for each $b \subseteq \mathrm{pcf}(a)$ of cardinality $\leqq \kappa$, $g_b \in \Pi b$ be defined by

$$g_b(\theta) = \sup(\theta \cap N_{\gamma(*)}).$$

Let $g = \bigcup\{g_b : b \subseteq \mathrm{pcf}(a), |b| \leqq \kappa\}$. By Fact 27(1)

$$g_a \equiv f^{\kappa,a}_{\lambda,g(\lambda)} \bmod J^{\kappa,a}_{<\lambda}.$$

We shall show that $g_a \in M$ thus deriving the desired contradiction [as $f \in N_{\gamma(*)}$, clearly $f \leqq g_a$].

For each $i < \sigma$, by Fact 27(4) there are $\theta_i(l) \in \mathrm{pcf}(a_i)$,

$$b_i(l) \subseteq a_i - \bigcup_{m<l} b_i(m), \qquad b_i(l) \equiv b^{\kappa,a}_{\theta_i(l)} \bmod J^{\kappa,a}_{<\theta_i(l)},$$

$$g_a \restriction b_i(l) = f^{\kappa,a}_{\theta_i(l),g(\theta_i(l))} \restriction b_i(l), \quad \text{and} \quad \theta_i(0) = \max \mathrm{pcf}(a_i), \quad \theta_i(l+1) < \theta_i(l),$$

and $\theta_i(l)$, $b_i(l)$ are defined just for $l \leqq n(i)$, and lastly

$$a_i = \bigcup_{l \leqq n(i)} b_i(l).$$

As $\mathscr{P}(\kappa) \subseteq M$, $\{\langle i, l, \theta_i(l)\rangle : i < \sigma, l < n(i)\} \in M$, it is enough to show that

$$\langle f^{\kappa, a_i}_{\theta_i(l), g(\theta_i(l))} : l \leqq n(i), i < \sigma\rangle \in M.$$

As $\langle\langle f^{\kappa,b}_{\theta,\alpha} : \alpha < \theta\rangle : b \subseteq a, \theta \in \mathrm{pcf}(b)\rangle \in M$ it is enough to have $g_c = g \restriction c \in M$ where $c = \{\theta_i(l) : i < \sigma, l < n(i)\}$. Now $|c| \leqq \sigma \leqq \kappa$, $c \subseteq \mathrm{pcf}(a)$ hence $c \in M$ and $\mathrm{pcf}(c) \subseteq \mathrm{pcf}(a)$ (so $c \subseteq R^{\kappa,2}$), therefore $\max \mathrm{pcf}(c) \leqq \max \mathrm{pcf}(a) = \lambda$. If $\max \mathrm{pcf}(c) = \lambda$ we finish. Otherwise (as $\mathrm{pcf}(c) \subseteq \mathrm{pcf}(a)$) $\max \mathrm{pcf}(c) \leqq \lambda(0)$, and as $c \in M$, by 27 easily $g_c \in M$, and as we said above, we finish.

29. Conclusion. (1) If $\lambda \in R^{\kappa,2}_{\mathrm{ac}}$, κ minimal, then

(i) for some a

$$\lambda = \max \mathrm{pcf}(a), \quad \sup a < \lambda, \quad a \in I^{\kappa,2},$$

(ii) for any such a, $J^{\kappa,a}_{<\lambda}$ is a κ-complete ideal on a, hence $[b \subseteq a \wedge |b| < \kappa \Rightarrow b \in J^{a,\kappa}_{<\lambda}]$ (and $\lambda = \mathrm{tcf}(\Pi a, \leqq_{J^{\kappa,a}_{<\lambda}})$ and necessarily is regular.

(2) If $(\forall \alpha < \delta)(\aleph^{\kappa}_{\alpha} < \aleph_{\delta})$, $\aleph_{\delta}$ is singular, $a \subseteq R^{\kappa,2} \cap \aleph_{\delta}$, $\kappa < \aleph_{\delta}$, $\mu \in \mathrm{pcf}(a) \cap [\aleph_{\delta}, \aleph^{\kappa}_{\delta})$, *then* there is $a \subseteq R^{\mathrm{cf}\,\delta,2} \cap \aleph_{\delta}$, $|a| \leqq \mathrm{cf}\,\delta$ such that $\mu \in \mathrm{pcf}(a)$ (even a has order type $\leqq \mathrm{cf}\,\delta$).

Proof. (1) (i) This holds by the definition of $R^{\kappa,2}_{\mathrm{ac}}$ and 17(5).

(ii) If $J^{\kappa,a}_{<\lambda}$ is not κ-complete, we can find $\sigma < \kappa$, $a_i \in J^{\kappa,a}_{<\lambda}$ for $i < \sigma$,

$$\bigcup_{i<\sigma} a_i \notin J^{\kappa,a}_{<\lambda}.$$

W.l.o.g. the a_i's are pairwise disjoint (otherwise use $a_i \overset{\mathrm{def}}{=} a_i - \bigcup_{j<1} a_j$) and $a = \bigcup_{i<\sigma} a_i$ (otherwise replace a by $a' \overset{\mathrm{def}}{=} \bigcup_{i<\sigma} a_i$).

By 28 we get a contradiction to minimality of κ.

(2) Easy (by 12 and 29(1)).

30. Main Lemma. *Suppose* $\lambda \in R^{\kappa,2}_{\mathrm{ac}}$, *$\sigma$ is minimal such that* $\lambda \in R^{\sigma,2}_{\mathrm{ac}}$.

(1) *For some* $a \in I^{\kappa,2} : a \subseteq \lambda$ *and* $\lambda \in \mathrm{pcf}(a)$.

(2) *For a as above for some* $b \subseteq a$, $\lambda = \max \mathrm{pcf}(b)$ *and* $\lambda > \max \mathrm{pcf}(b \cap \mu)$ *for* $\mu \in b$ *(take first* $b = b^{\kappa,a}_{\lambda}$ *and then* $b \cap \mu$ *for the minimal* $\mu \in b$ *such that* $\lambda \in \mathrm{pcf}(b \cap \mu)$).

(3) *For b as above* $J^{\kappa,b}_{<\lambda}$ *is σ-complete.*

(4) *For* b *as above for some* $c \subseteq b$, *letting* $d_1 \stackrel{\text{def}}{=} \text{pcf}(c)$, $\lambda = \max d_1$, $d \stackrel{\text{def}}{=} d_1 - \{\lambda\}$ *has no last element*; *in fact*

$$d - \theta = \text{pcf}(b) - \{\mu : \mu \in \text{pcf}(b), \mu < \lambda, \text{pcf}(b) - \mu \textit{ is finite}\} - \theta$$

for some $\theta \in d$ (*so* $\text{pcf}(b) - \text{pcf}(c)$ *is finite and* $J^{\kappa,c}_{<\lambda}$ *is* σ-*complete by* (3)).

(5) *For* c *as above, the order type of the sets* c, d *has cofinality* $\geqq \sigma$.

(6) *Letting* J^{bd}_d *be the ideal of bounded subsets of* d, then $\text{tcf}(\Pi d, <_{J^{\text{bd}}_d}) = \lambda$.

(7) *For some* $d_2 \subseteq d$ *of order type* $\text{cf}(\text{otp } d)$, $\text{pcf}_{|d_2|}(d_2) = \{\lambda\}$.

REMARK. The main conclusion here is part (6).

PROOF. (1) By definition of $R^{\kappa,2}_{\text{ac}}$.

(2) Clear.

(3) By the assumption on σ and 28.

(4) Clearly $\{\mu \in \text{pcf}(b) : \mu < \lambda, \text{pcf}(b) = \mu \text{ is finite}\}$ is finite. So by 16

$$c \stackrel{\text{def}}{=} b - \bigcup\{b^{\kappa,b}_\mu : \mu \in \text{pcf}(b), \mu < \lambda, |\text{pcf}(b) - \mu| < \aleph_0\}$$

is as required.

(5) By (3) $J^{\kappa,c}_{<\lambda}$ is σ-complete. If $\mu_i \in d$ for $i < \delta < \sigma$, then $b^{\kappa,c}_{\mu_i} \in J^{\kappa,c}_{<\lambda}$, so $\bigcup_{i<\delta} b^{\kappa,c}_{\mu_i} \in J^{\kappa,c}_{<\lambda}$ hence by 16 for some $n < \omega$,

$$\theta_0, \ldots, \theta_{n-1} \in d_1 = \text{pcf}(c) - \{\lambda\}, \qquad \bigcup_{i<\delta} b^{\kappa,c}_{\mu_i} \subseteq \bigcup_{l<n} b^{\kappa,c}_{\theta_l}.$$

So necessarily each μ_i is $\leqq \text{Max}\{\theta_l : l < n\}$, so (as d has no last element by 30(4)) $\sup_{i<\delta} \mu_i < \sup d$, as required.

As for $\text{cf}(\text{otp}(c))$, use 30(2) and σ-completeness of the ideal.

(6) For each $\alpha < \lambda$, choose, if possible, a function $g = g_\alpha$, $g \in \Pi d$ such that: for every $\theta \in \text{pcf}(c)$ (for $\theta = \lambda$ stipulating $g(\theta) = \alpha$)

$$g \restriction b^{\kappa,c}_\theta \equiv f^{\kappa,c}_{\theta,g(\theta)} \bmod J^{\kappa,c}_{<\theta}.$$

Let $A = \{\alpha < \lambda : g_\alpha \text{ is well defined}\}$. Now

FACT α. A is an unbounded subset of λ.

PROOF. By 27.

FACT β. For $\alpha < \beta$ in A, $g_\alpha < g_\beta \bmod J^{\text{bd}}_d$.

PROOF OF FACT β. Clearly $f^{\kappa,c}_{\lambda,\alpha} < f^{\kappa,c}_{\lambda,\beta} \bmod J^{\kappa,c}_{<\lambda}$ hence

$$c_1 = \{\theta \in c : f^{\kappa,c}_{\lambda,\alpha}(\theta) \geqq f^{\kappa,c}_{\lambda,\beta}(\theta)\} \in J^{\kappa,c}_{<\lambda}.$$

Now $J^{\kappa,c}_{<\lambda}=\bigcup\{J^{\kappa,c}_{\leqq\mu}:\mu\in\mathrm{pcf}(c),\mu\neq\lambda\}=\bigcup\{J^{\kappa,c}_{<\mu}:\mu\in\mathrm{pcf}(c),\mu\neq\lambda\}$ (as $\mathrm{pcf}(c)-\{\lambda\}$ has no last element). So for some $\theta(*)\in\mathrm{pcf}(c)-\{\lambda\}$, $c_1\in J^{\kappa,c}_{<\theta(*)}$. Also

$$c_2=\{\mu\in c:f^{\kappa,c}_{\lambda,\alpha}(\mu)\neq g_\alpha(\mu)\}\in J^{\kappa,c}_{<\lambda},$$

$$c_3=\{\mu\in c:f^{\kappa,a}_{\lambda,\beta}(\mu)\neq g_\beta(\mu)\}\in J^{\kappa,c}_{<\lambda}.$$

So w.l.o.g. also c_2, $c_3\in J^{\kappa,c}_{<\theta(*)}$. Now

$$g_\alpha\restriction(c-c_1\cup c_2\cup c_3)=f^{\kappa,c}_{\lambda,\alpha}\restriction(c-c_1\cup c_2\cup c_3)$$
$$<f^{\kappa,c}_{\lambda,\beta}\restriction(c-c_1\cup c_2\cup c_3)=g_\beta\restriction(c-c_1\cup c_2\cup c_3).$$

Hence for every $\theta\in d$, if $\theta\geqq\theta(*)$ (as $c_1\cup c_2\cup c_3\in J^{\kappa,c}_{<\theta(*)}\subseteq J^{\kappa,c}_{<\theta}$) *then*

$$f^{\kappa,c}_{\theta,g_\alpha(\theta)}\restriction b^{\kappa,c}_\theta\equiv g_\alpha\restriction b^{\kappa,c}_\theta<g_\beta\restriction b^{\kappa,c}_\theta\equiv f^{\kappa,c}_{\theta,g_\beta(\theta)}\bmod J^{\kappa,c}_{<\theta}.$$

Hence by the choice of $\langle f^{\kappa,c}_{\theta,\gamma}:\gamma<\theta\rangle$ necessarily $g_\alpha(\theta)<g_\beta(\theta)$ as required.

FACT γ. If $g\in\Pi d$, then for some $\alpha\in A$, $g<g_\alpha\bmod J^{\mathrm{bd}}_d$ [use 27 with $g\in N_0$].

By Facts α, β, γ we finish.

(7) Follows by 30(6).

§5. Applications

31. DEFINITION. $\mathrm{Pr}_0(\lambda,\kappa,\theta)$ where $\lambda\geqq\kappa+\theta$, λ an infinite cardinal, means that there is a two place function c from λ to κ which witnesses it, which means:

(*) *if* $\xi<\theta$ and for $i<\lambda$, $\langle\alpha_{i,\zeta}:\zeta<\xi\rangle$ is a strictly increasing sequence of ordinals $\geqq i$, $<\lambda$ and h is a two place function from $\xi=\{\zeta:\zeta<\xi\}$ to κ, *then* there are $i<j<\lambda$, such that $\bigcup_{\zeta<\xi}\alpha_{i,\zeta}<\alpha_{j,0}$ and

$$[\zeta_1<\xi\wedge\zeta_2<\xi\Rightarrow c(\alpha_{i,\zeta_1},\alpha_{j,\zeta_2})=h(\zeta_1,\zeta_2)].$$

32. DEFINITION. (1) $\mathrm{Pr}_1(\lambda,\kappa,\theta)$ where $\lambda\geqq\kappa+\theta$, λ an infinite cardinal, means that there is a two place function c from λ to κ which witnesses it, which means:

(*) *if* $\xi<\theta$ and for $i<\lambda$, $\langle\alpha_{i,\zeta}:\zeta<\xi\rangle$ is a strictly increasing sequence of ordinals $\geqq i$, $<\lambda$ and $\gamma<\kappa$, *then* there are $i<j<\lambda$ such that $\bigcup_{\zeta<\xi}\alpha_{i,\zeta}<\alpha_{j,0}$ and

$$[\zeta_1<\xi\wedge\zeta_2<\xi\Rightarrow c(\alpha_{i,\zeta_1},\alpha_{j,\zeta_2})=\gamma].$$

(2) $\mathrm{Pr}_1^-(\lambda, \kappa, \theta)$ is defined similarly but we add to the assumption of (∗):

$$[i < \lambda, \zeta_1 < \zeta_2 < \xi \Rightarrow c(\alpha_{i,\zeta_1}, \alpha_{i,\zeta_2}) = \gamma].$$

33. Definition. (1) $\mathrm{Pr}_2(\lambda, \kappa, \theta)$ where $\lambda \geqq \kappa + \theta$, λ an infinite cardinal, means that there is a symmetric two place function c from λ to κ which witnesses it, which means:

(∗) *if* $\xi < \theta$ and for $i < \lambda$, $\langle \alpha_{i,\zeta} : \zeta < \xi \rangle$ is a strictly increasing sequence of ordinals $\geqq i$ and $< \lambda$ and $\gamma < \kappa$, *then* there are $i < j < \lambda$, such that $\bigcup_{\zeta < \xi} \alpha_{i,\zeta} < \alpha_{j,0}$ and for $\zeta_1, \zeta_2 < \xi$

(i) $\zeta_1 = \zeta_2 \Rightarrow c(\alpha_{i,\zeta_1}, \alpha_{j,\zeta_2}) = \gamma$,

(ii) $\zeta_1 \neq \zeta_2 \Rightarrow c(\alpha_{i,\zeta_1}, \alpha_{j,\zeta_2}) = c(\alpha_{i,\zeta_1}, \alpha_{i,\zeta_2})$.

(2) $\mathrm{Pr}_2^a(\lambda, \kappa, \theta)$ is defined similarly, replacing (i), (ii) by

(i)′ $\xi > \zeta_1 > \zeta_2 \Rightarrow c(\alpha_{i,\zeta_1}, \alpha_{j,\zeta_2}) = 2\gamma$,

(ii)′ $\zeta_1 < \zeta_2 < \xi \Rightarrow c(\alpha_{i,\zeta_1}, \alpha_{j,\zeta_2}) = c(\alpha_{i,\zeta_1}, \alpha_{i,\zeta_2})$

(iii)′ $\zeta < \xi \Rightarrow c(\alpha_{i,\zeta}, \alpha_{j,\zeta}) = 2\gamma + 1$.

(3) $\mathrm{Pr}_2^b(\lambda, \kappa, \theta)$ is defined similarly, interchanging the conclusions of (i)′, (ii)′.

(4) $\mathrm{Pr}_2(\lambda, \kappa, \theta, \mu)$ is defined as in (1), but we add to the assumption of (∗) an ordinal $\gamma_0 < \mu$ and get a set $u \subseteq \lambda$ of order type γ_0 such that any $i < j$ from u satisfies the conclusion of (∗).

34. Definition. (1) $\mathrm{Pr}_3^S(\lambda, \kappa, \theta)$ where $\lambda \geqq \kappa + \theta$, λ an infinite cardinal, means that there is a two place function c from λ to κ which witnesses it, which means:

(∗) *if* $\xi < \theta$ and for $i < \lambda$, $\langle \alpha_{i,\zeta} : \zeta < \xi \rangle$ is a strictly increasing sequence of ordinals $\geqq i$ and $< \lambda$ and $2\gamma + 1 < \kappa$ and $\zeta(*) < \xi$, *then* there are $i < j < \lambda$ such that $\bigcup_{\zeta < \xi} \alpha_{i,\zeta} < \alpha_{j,0}$ and

(i) $\zeta < \zeta(*) \Rightarrow c(\alpha_{i,\zeta(*)}, \alpha_{j,\zeta}) = 2\gamma$,

(ii) $\zeta(*) < \zeta < \xi \Rightarrow c(\alpha_{i,\zeta(*)}, \alpha_{j,\zeta}) = c(\alpha_{i,\zeta(*)}, \alpha_{j,\zeta})$,

(iii) $c(\alpha_{i,\zeta(*)}, \alpha_{j,\zeta(*)}) = 2\gamma + 1$.

(2) $\mathrm{Pr}_3^L(\lambda, \kappa, \theta)$ is defined similarly replacing (i), (ii), (iii) by:

(i)′ $\zeta < \zeta(*) \Rightarrow c(\alpha_{i,\zeta}, \alpha_{j,\zeta(*)}) = c(\alpha_{i,\zeta}, \alpha_{i,\zeta(*)})$,

(ii)′ $\zeta(*) < \zeta < \xi \Rightarrow c(\alpha_{i,\zeta}, \alpha_{j,\zeta(*)}) = 2\gamma$,

(iii)′ $c(\alpha_{i,\zeta(*)}, \alpha_{j,\zeta(*)}) = 2\gamma + 1$.

35. Definition. In Definitions 31–34, omitting θ means that it is $\aleph_0$.

36. Observation. (1) $\mathrm{Pr}_0(\lambda,\kappa,\theta)$ *implies* $\mathrm{Pr}_1(\lambda,\kappa,\theta)$, $\mathrm{Pr}_2(\lambda,\kappa,\theta)$.

(2) $\mathrm{Pr}_2^y(\lambda,\kappa,\theta)$ implies $\mathrm{Pr}_3^x(\lambda,\kappa,\theta)$ when $y=a$, $x=S$ or $y=b$, $x=L$.

(3) $\mathrm{Pr}_1(\lambda,\kappa,\theta)$ implies $\mathrm{Pr}_1^-(\lambda,\kappa,\theta)$.

(4) $\mathrm{Pr}_2(\lambda,\kappa,\theta)$ implies $\mathrm{Pr}_1^-(\lambda,\kappa,\theta)$.

(5) $\mathrm{Pr}_2(\lambda,\kappa,\theta,2)$ is equivalent to $\mathrm{Pr}_2(\lambda,\kappa,\theta)$.

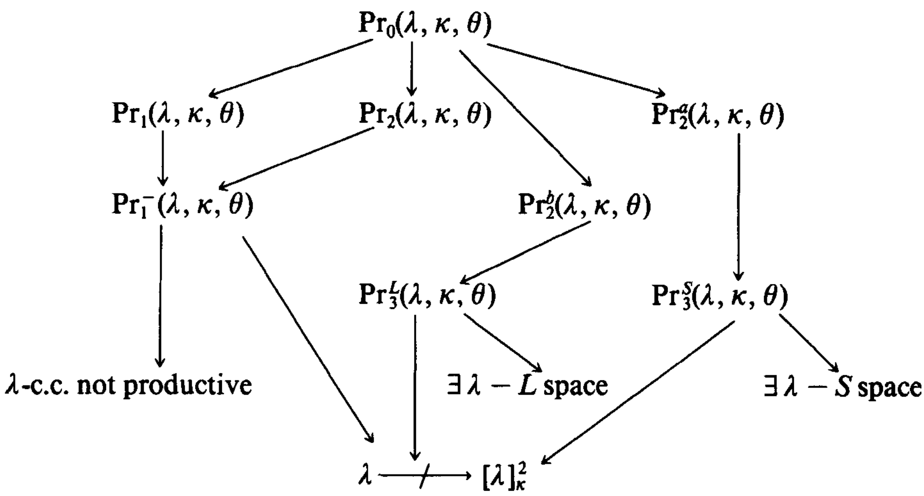
(6) If $\kappa_1\leqq\kappa_2$, $\theta_1\leqq\theta_2$, $\mu_1\leqq\mu_2$ *then* $\mathrm{Pr}_0(\lambda,\kappa_2,\theta_2)\Rightarrow\mathrm{Pr}_0(\lambda,\kappa_1,\theta_1)$, $\mathrm{Pr}_1(\lambda,\kappa_2,\theta_2)\Rightarrow\mathrm{Pr}_1(\lambda,\kappa_1,\theta_1)$, $\mathrm{Pr}_1^-(\lambda,\kappa_2,\theta_2)\Rightarrow\mathrm{Pr}_1^-(\lambda,\kappa_1,\theta_1)$, $\mathrm{Pr}_2(\lambda,\kappa_2,\theta_2,\mu_2)\Rightarrow\mathrm{Pr}_2(\lambda,\kappa_1,\theta_1,\mu_1)$, $\mathrm{Pr}_3(\lambda,\kappa_2,\theta_2)\Rightarrow\mathrm{Pr}_3(\lambda,\kappa_1,\theta_1)$.

(7) $\mathrm{Pr}_1^-(\lambda,2)$ implies that the λ-c.c. is not productive for Boolean algebras (see [Sh 7] proof of 1.10).

(8) $\mathrm{Pr}_3^X(\lambda,2)$ implies that there are $\lambda-X$ spaces for $X=L,S$ (Hausdorf with a basis of clopen sets). (See [Sh 7] proof of 1.10.)

(9) $\mathrm{Pr}_1^-(\lambda,n)$ implies that there are Boolean algebras $B_l\,(l<n)\cdot\Pi_{l<n}B_l$ does not satisfy the λ-c.c. but $\Pi_{l<n,l\neq m}B_l$ satisfies it for $m<n$. (See [Sh 7] proof of 1.10.)

37. Diagram. (For $\kappa\geqq2$, $\theta\geqq2$.)



38. Lemma. (1) *Suppose* $c\subseteq R^0$, $\mathrm{tcf}(\Pi c,\ \leqq_{J_c^{bd}})=\lambda$ (*c has no last element*) $(\forall\mu\in c)(\mu<\lambda)$ *and let* $\chi=\mathrm{Min}\{|c-\mu|:\mu\in c\}<\sup c$. *Then* $\mathrm{Pr}_1(\lambda,\chi,\mathrm{cf}(c))$.

(2) *If in addition for arbitrarily large* $\theta\in c$, $\mathrm{Pr}_l(\theta,\chi',2\sigma)$, $\theta^{<\sigma}<\mathrm{Min}(c-\theta^+)$, $l\in\{0,1\}$, $\sup(c\cap\theta)\in c$, $\sigma\leqq\mathrm{cf}(c)$ then $\mathrm{Pr}_l(\lambda,\chi',\sigma)$.

39. REMARK. (1) Every unbounded $c' \subseteq c$ satisfies all the assumptions (though: maybe with a smaller χ). So w.l.o.g. $|c| = \chi$.

(2) If in 38(2) we want to interpret σ as order type, 2σ should be replaced by $\sigma 2$.

(3) In 38(1) we can have $\mathrm{Pr}_i(\theta, \chi'_\theta, 2\sigma_\theta)$ for $\theta \in c$, such that

$$[\mathrm{v}(1) \leqq \theta(2) \Rightarrow \chi'_{\theta(1)} \leqq \chi'_{\theta(2)} \wedge \sigma_{\theta(1)} \leqq \sigma_{\theta(2)}]$$

and get $\mathrm{Pr}_i(\lambda, \Sigma_\theta \chi'_\theta, \Sigma_\theta \sigma_\theta)$.

(4) $\mathrm{cf}(c)$ is the cofinality of the order type of c.

PROOF. (1) Let $\langle f_\alpha : \alpha < \lambda \rangle$ exemplify $\mathrm{tcf}(\Pi c, \leqq_{J_c^{\mathrm{bd}}}) = \lambda$, $\langle c_i : i < \chi \rangle$ a partition of c to χ sets, each c_i is an unbounded subset of c. Let $h : c \to \chi$ be such that $\theta \in c_{h(\theta)}$.

Let us define two two-place functions, θ, e from λ with range of power χ : for $\alpha < \beta < \lambda$, $\theta(\alpha, \beta) = \sup\{\theta : f_\alpha(\theta) \geqq f_\beta(\theta)\}$ (so if there is a maximal θ for which $f_\alpha(\theta) \geqq f_\beta(\theta)$, it is $\theta(\alpha, \beta)$) and

$$e(\alpha, \beta) = h(\theta(\alpha, \beta)).$$

Suppose $\xi < \mathrm{cf}(c)$, $\langle \alpha_{\beta,\zeta} : \zeta < \xi \rangle$ is a strictly increasing sequence of ordinals $\geqq \beta$ but $< \lambda$ for each $\beta < \lambda$. Now for any given $i(*) < \chi$ we should find $\beta < \gamma < \lambda$ such that $\bigcup_\zeta \alpha_{\beta,\zeta} < \alpha_{\gamma,0}$ and for every $\zeta_1, \zeta_2 < \xi$, $\theta(\alpha_{\beta,\zeta_1}, \alpha_{\gamma,\zeta_2})$ belongs to $c_{i(*)}$.

Let χ^* be a regular large enough cardinal. Let M_0 be an elementary submodel of $(H(\chi^*), \in, <^*)$ where $<^*$ is a well ordering of $H(\chi^*)$, to which λ, c, ξ, $\langle \langle \alpha_{\beta,\zeta} : \zeta < \xi \rangle : \beta < \lambda \rangle$, $\langle f_\alpha : \alpha < \lambda \rangle$ belongs, $\xi \cup c \subseteq M$, and $\| M_0 \| < \sup c$.

Let $c' = \{\theta \in c : \sup(M_0 \cap \theta) < \theta\}$, so $c' \equiv c \bmod J_c^{\mathrm{bd}}$. Define a function $g \in \Pi c : g(\theta)$ is $\sup(M_0 \cap \theta)$ for $\theta \in c'$ and zero otherwise. As $\langle f_\beta : \beta < \lambda \rangle$ exemplify $\mathrm{tcf}(\Pi c, <_{J_c^{\mathrm{bd}}}) = \lambda$ for some $\beta(0) < \lambda$, $g < f_{\beta(0)} \bmod J_c^{\mathrm{bd}}$, and $\beta(0) > \sup(M_0 \cap \lambda)$.

As $\alpha_{\beta(0),\zeta} \geqq \beta(0)$ for each $\zeta < \xi$ for some $\theta_\zeta^0 \in c$,

$$[\theta_\zeta^0 < \theta \in c \Rightarrow g(\theta) < f_{\alpha_{\beta(0),\zeta}}(\theta)].$$

Let $\theta(0) = \sup\{\theta_\zeta^0 : \zeta < \xi\}$ so as $\mathrm{cf}(c) > \xi$ clearly $\theta(0) < \sup(c)$. Let $\theta(1) \in c_{i(*)}$ be $> \theta(0)$. Let for $\beta < \lambda$, $f_\beta^* \in \Pi c$ be defined by

$$f_\beta^*(\theta) = \mathrm{Min}\{f_{\alpha_{\beta,\zeta}}(\theta) : \zeta < \xi\}.$$

Easily $f_\beta \leqq f_\beta^* \bmod J_c^{\mathrm{bd}}$ (as $\mathrm{cf}(\mathrm{otp}\, c) > \xi$). Let

$$c^* = \{\theta \in c : \sup\{f_\beta^*(\theta) : \beta < \lambda\} = \theta\}.$$

So $c^* = c \bmod J_c^{bd}$ [otherwise define $g^* \in \Pi c : g^*(\theta) = \sup\{f_\beta^*(\theta) : \beta < \lambda\}$ if $\theta \in c - c^*$, and zero otherwise. So for some $\beta < \lambda$, $g^* < f_\beta \bmod J_c^{bd}$, and we get a contradiction easily].

So we could have chosen $\theta(1) \in c^*$, such that $\theta(1) > \| M_0 \|$ and for some $\beta(1)$,

$$\gamma \overset{\text{def}}{=} f^*_{\beta(1)}(\theta(1)) > \sup\{f_{\alpha_{\beta(0),\zeta}}(\theta(1)) : \zeta < \xi\}.$$

Let M_1 be the Skolem Hull of $M \cup \{\gamma\}$ (in $(H(\chi^*), \in, <^*)$). Now clearly $(H(\chi^*), \in, <^*) \models (\exists \beta < \lambda) f^*_\beta(\theta(1)) = \gamma$; as λ, $\langle f^*_\beta, \beta < \lambda \rangle$, $\theta(1)$, γ are in M_1 there is $\beta(2) \in M_1 \cap \lambda$, $f^*_{\beta(2)}(\theta(1)) = \gamma$. So

$$[\zeta_1, \zeta_2 < \xi \Rightarrow f^*_{\alpha_{\beta(0),\zeta_1}}(\theta(1)) < f^*_{\alpha_{\beta(2),\zeta_2}}(\theta(1))].$$

Easily, for every regular cardinal $\sigma \in M_0$, if $\sigma > \theta(1)$, then $\sup(M_0 \cap \sigma) = \sup(M_1 \cap \sigma)$.

As $\beta(0) > \sup(M_0 \cap \lambda)$, also $\beta(0) > \sup(M_1 \cap \lambda)$, but $\beta(2) \in M_1$, so $\beta(0) > \beta(2)$, and similarly $\alpha_{\beta(2),\zeta_1} < \alpha_{\beta(0),\zeta_2}$ (for $\zeta_1, \zeta_2 < \xi$). Also for every $\theta \in c$, if $\theta > \theta(1)$, $\zeta_1, \zeta_2 < \xi$, then $f_{\alpha_{\beta(2),\zeta_1}}(\theta) \in M_1$ hence $f_{\alpha_{\beta(2),\zeta_1}}(\theta) < \sup(M_0 \cap \theta)$ hence $f_{\alpha_{\beta(2),\zeta_1}}(\theta) < g(\theta)$, but $g(\theta) < f_{\alpha_{\beta(0),\zeta_2}}(\theta)$. So $\theta(\alpha_{\beta(2),\zeta_1}, \alpha_{\beta(2),\zeta_2}) = \theta(1)$, but $h(\theta(1)) = i(*)$, so we finish.

(2) Let c_θ be a two place function from θ to χ', which exemplifies our hypothesis when possible. Let

$$c^*(\alpha, \beta) = c_{\theta(\alpha,\beta)}(f_\alpha(\theta(\alpha, \beta)), f_\beta(\theta(\alpha, \beta))).$$

The rest is left to the reader.

§6. Additional information

40. Lemma. *Suppose* $a \in I^{\kappa,2}$, $\lambda = \max \mathrm{pcf}(a)$, $c = \mathrm{pcf}(a) - \{\lambda\}$, *$c$ has no last element*, $\chi = \min\{|c - \mu| : \mu \in c\}$. *Then* $\mathrm{Pr}_2(\lambda, \chi, \aleph_0)$.

Proof. Let $\langle c_i : i < \lambda \rangle$ be a partition of c such that each c_i is an unbounded subset of c. Let $h : c \to \chi$ be defined by $\theta \in c_{h(\theta)}$. W.l.o.g. $(\forall \mu \in c)\mu > (2^\kappa)^+$. Let $f_\alpha \overset{\text{def}}{=} f^{k,a}_{\lambda,\alpha}$, $\sigma = \aleph_0$. Let us define two two-place functions θ^-, e^- from λ to $\mathrm{cf}\,\chi$ respectively:

$$\text{for } \alpha < \beta < \lambda, \quad \theta^-(\alpha, \beta) \overset{\text{def}}{=} \min\{\mu : \{\theta \in a : f_\alpha(\theta) = f_\beta(\theta)\} \in I^{\kappa,2}_{<\mu^+}\}$$

(so $\theta^-(\alpha, \beta) \in \mathrm{pcf}(a)$ and as $f_\alpha \neq f_\beta \bmod I^{\kappa,2}_{<\lambda}$ clearly $\theta^-(\alpha, \beta) \in c$. Let $e^-(\alpha, \beta) \overset{\text{def}}{=} h(\theta^-(\alpha, \beta))$.

Define A as in the proof of 30(6). W.l.o.g. we restrict ourselves to A.

Suppose $\xi < \aleph_0$, $\langle \alpha_{\beta,\zeta} : \zeta < \xi \rangle$ is a strictly increasing sequence of ordinals $\geqq \beta$ but from A, for each $\beta < \lambda$.

Now for any given $i(*) < \chi$ we should find $\beta < \gamma < \lambda$ such that $\bigcup_\zeta \alpha_{\beta,\zeta} < \alpha_{\gamma,0}$ and $\theta(\alpha_{\beta,\zeta_1}, \alpha_{\gamma,\zeta_2})$ belongs to $c_{i(*)}$ if $\zeta_1 = \zeta_2$ and is $\theta^-(\alpha_{\beta,\zeta_1}, \alpha_{\beta,\zeta_2})$ if $\zeta_1 \neq \zeta_2$.

Let χ^* be regular large enough. Let M_0 be an elementary submodel of $(H(\chi^*), \in, <^*)$ where $<^*$ is a well ordering of $H(\chi^*)$, such that the following belongs to M_0: $\lambda, a, c, \langle \langle \alpha_{\beta,\zeta} : \zeta < \xi \rangle : \beta < \lambda \rangle, \langle f_\alpha : \alpha < \lambda \rangle, \mu$ for $\mu \in c$, and M has cardinality 2^κ and every subset of M_0 of cardinality $\leqq \kappa$ belongs to M.

Let $g \in \Pi c$ be defined by $g(\theta) = \sup(M_0 \cap \theta)$; as $a \subseteq R^{0,\kappa}$, $(\forall \mu \in a)[\mu > (2^\kappa)^+]$, clearly $g \in \Pi c$; so for some $\beta(0) \in A$, $g \restriction a < f_{\beta(0)} \bmod J^{\kappa,a}_{<\lambda}$. But $J^{\kappa,a}_{<\lambda} = \bigcup_{\theta \in c} J^{\kappa,a}_{<\theta}$, so

$$a^\zeta_1 \stackrel{\text{def}}{=} \{\theta \in a : g(\theta) \geqq f_{\alpha_{\beta(0),\zeta}}(\theta)\}$$

belongs to $J^{\kappa,a}_{<\theta_\zeta}$, for some θ_ζ. As $J^{\kappa,a}_{<\lambda}$ is σ-complete, $\xi < \sigma$, for some $\theta(0) \in c$, $\bigcup_{\zeta<\xi} a^\zeta_1 \in J^{\kappa,a}_{<\theta(0)}$. Let $\theta(1) \in c_{i(*)}$, $\theta(1) > \theta(0)$.

As $\beta(0) \in A$, for $\zeta < \xi$ we can find $\mu_{\zeta,l}$ $(l \leqq n_\zeta)$ from $c \cap \theta(1)^+$ and partition $\langle a_{\zeta,l} : l \leqq n_\zeta \rangle$ of $a_2 = \bigcup_{\zeta<\xi} a^\zeta_1 \cup b^{\kappa,a}_{\theta(1)} \cup \bigcup \{b^\kappa_\theta : \theta = (\alpha_{\beta(0),\xi_1}, \alpha_{\beta(0),\xi_2})\}$ such that:

$$f_{\alpha_{\beta(0),\zeta}} \restriction a_2 = \bigcup_{l \leqq n_\zeta} f^{\kappa,a}_{\mu_{\zeta,l}, \gamma_{\zeta,l}} \restriction {}_{\zeta,l} \qquad \text{where } \gamma_{\zeta,l} = f_{\alpha_{\beta(0),\zeta}}(\mu_{\zeta,l}).$$

Let M_2 be the Skolem Hull of

$$M_1 \cup \{\langle \langle \langle \mu_{\zeta,l}, f_{\alpha_{\beta(0),\zeta}}(\mu_{\zeta,l}), a_{\zeta,l} \rangle : l \leqq n_\zeta \rangle : \zeta < \xi \rangle\}.$$

So there is $\beta(1) \in M_1$, $\beta(1) \in A$, and $f_{\alpha_{\beta(1),\zeta}} \restriction a_2 = f_{\alpha_{\beta(0),\zeta}} \restriction a_2$. As in the proof of 38, for $\theta \in \mathrm{pcf}(a) - \theta(1)^+$,

$$\sup(M_0 \cap \theta) = \sup(M_2 \cap \theta).$$

Easily $\beta(1) < \beta(0)$ are as required.

41. Remark. Can we have $\mathrm{Pr}_2(\lambda, \chi', \sigma)$? Yes, if e.g. for some $\mu_0 \in c$, $[\mu_0 < \mu \in c \Rightarrow \mu^\sigma < \mathrm{Min}(c - \mu^+)]$, even replacing c by $c' \subseteq c$, $\sup c = \sup c'$, $\chi' = \mathrm{Min}\{|c' - \mu| : \mu \in c\}$.

42. Conclusion. If $\lambda \in R^{\kappa,2}_{\mathrm{ac}}$, κ minimal, then $\mathrm{Pr}_1(\lambda, \kappa, \kappa)$ and $\mathrm{Pr}_2(\lambda, \kappa, \aleph_0)$.

PROOF. By 30(6), (5) the assumption of 38 and 40 holds. By 38 and 40 we get respectively our two conclusions.

43. CONCLUSION. (1) If $\lambda > 2^{\mathrm{cf}\,\lambda}$ then for some κ,

$$\mathrm{Pr}_1(\lambda^+, \kappa, \kappa), \quad \mathrm{Pr}_2(\lambda^+, \kappa, \aleph_0)$$

(and see 37).

(2) If $\mu < \lambda \leqq \mu^\kappa$, $\mathrm{cf}\,\mu > \aleph_0$, $(\forall \theta < \mu)\theta^\kappa < \mu$, $\mathrm{cf}\,\lambda = \lambda$, then $\mathrm{Pr}_1(\lambda, \kappa, \kappa)$, $\mathrm{Pr}_2(\lambda, \kappa, \kappa)$.

44. LEMMA. *Suppose* $a \in I^{\kappa,2}$, $a \subseteq R^{\kappa,2}$, $\lambda = \max \mathrm{pcf}(a) > \sup(a)$.

$$[\forall \theta \in a][\forall \mu < \theta][\mu^{<\sigma} < \theta],$$

$a \cap \theta \in J^{\kappa,a}_{<\lambda}$ *for* $\theta \in a$. *Suppose* μ_θ, σ_θ *are cardinals for each* $\theta \in a$, *and*

$$[\theta_1 < \theta_2 \textit{ from } a \Rightarrow \mu_{\theta(1)} \leqq \mu_{\theta(2)} \wedge \sigma_{\theta(1)} \leqq \sigma_{\theta(2)}], \quad \sigma_\theta \leqq \sigma.$$

Then for $l = 2$, $\bigwedge_{\theta \in a} \mathrm{Pr}_l(\theta, \mu_\theta, \sigma_\theta)$ *implies* $\mathrm{Pr}_l(\lambda, \sup \mu_\theta, \sup \sigma_\theta)$.

PROOF. Straightforward.

REMARK. The improvement 2σ below enables us to have $\mathrm{Pr}_2(\lambda, 2)$ even when $I^{\kappa,a}_{<\lambda}$ is a maximal ideal.

45. LEMMA. *Suppose* $a \in I^{\kappa,2}$, $a \subseteq R^{\kappa,a}_{\mathrm{ia}}$, $\lambda = \max \mathrm{pcf}(a) > \sup(a)$, $\langle a_i : i < \sigma_1 \rangle$ *are pairwise disjoint subsets of* a, $a_i \notin J^{\kappa,a}_{<\lambda}$, $J^{\kappa,a}_{<\lambda}$ *is* σ_2-*complete and for* $\theta \in a$, $[\forall \mu < \theta]\mu^{<\sigma_2} < \theta$. *Then*

(1) $\mathrm{Pr}_2(\lambda, \sigma_1, \sigma_2)$.

Moreover

(2) $\mathrm{Pr}_2(\lambda, 2\sigma_1, \sigma_2)$.

(3) *If for* $\theta \in a$, $\mathrm{Pr}_2(\theta, \chi_\theta, \sigma_\theta)$, $\sigma_\theta \leqq \sigma_2$,

$$[\theta(1) \leqq \theta(2) \Rightarrow \chi_{\theta(1)} \leqq \chi_{\theta(2)} \wedge \sigma_{\theta(1)} \leqq \sigma_{\theta(2)}]$$

then $\mathrm{Pr}_2(\lambda, \Sigma_{\theta \in a} \chi_\theta, \Sigma_{\theta \in a} \sigma_\theta)$.

PROOF. (1) Let f_α be such that $(*)$ of 25 holds (and the other demands there). We define for $\alpha < \beta < \lambda$: $\theta(\alpha, \beta) = \mathrm{Min}\{\theta \in a : f_\alpha(\theta) \neq f_\beta(\theta)\}$. Now $\langle a_i : i < \sigma_1 \rangle$ is a partition of a, and let $h : a \to \sigma_1$ be defined by $\theta \in a_{h(\theta)}$.

Let for $\alpha < \beta < \lambda$, $e(\alpha, \beta) = h(\theta(\alpha, \beta))$. Let $\xi < \sigma_2$, $\langle \alpha_{\beta,\zeta} : \zeta < \xi \rangle$ be a strictly increasing sequence of ordinals $< \lambda$, $\geqq \beta$, and let $i(*) < \sigma_1$. Let us define $f^*_\beta \in \Pi a$ by

$$f^*_\beta(\theta) = \mathrm{Min}\{f_{\alpha_{\beta,\zeta}}(\theta) : \zeta < \xi\}.$$

As $\xi < \sigma_2$, $J^{\kappa,a}_{<\lambda}$ is σ_2-complete, clearly $f_\alpha \leqq f^*_\beta$ mod $J^{\kappa,a}_{<\lambda}$. Let

$$c^* = \{\theta \in a : \sup\{f^*_\beta(\theta) : \beta < \lambda\} = \theta\}.$$

Then $c^* \equiv c \bmod J^{\kappa,a}_{<\lambda}$.

So we can find $\theta^* \in c^* \cap c_i$ so there are $\beta(i) < \lambda$ for $i < \theta^*$ such that $f^*_{\beta(i)}(\theta^*) \geqq i$, w.l.o.g. $[i < j < \theta^* \Rightarrow \beta(i) < \beta(j)]$, and (by $(*)$ of 25) w.l.o.g. $\langle f_{\alpha_{\beta(i),\zeta}} \restriction (a \cap \theta^* : \zeta < \xi\rangle$ is the same for all. The rest should be clear.

(2) What for 2σ colours? Let $\mathrm{pr}(\alpha, \beta)$ be 0 if $\alpha < \beta \equiv f_\alpha(\theta(\alpha, \beta)) < f_\beta(\theta(\alpha, \beta))$, and 1 otherwise, and let $e''(\alpha, \beta) = \sigma_1 \times \mathrm{pr}(\alpha, \beta) + h(\theta(\alpha, \beta))$.

46. Claim. Suppose $\lambda = \max \mathrm{pcf}(a)$, $a \in I^{\kappa,2}$, $J^{\kappa,a}_\lambda$ is σ-complete. Then $\mathrm{Pr}^-_1(\lambda, 2, \sigma)$.

Proof. For $\alpha < \beta$

$$c(\alpha, \beta) = \begin{cases} 0 & (\forall \mu \in a) f^{\kappa,a}_{\lambda,\alpha}(\mu) \leqq f^{\kappa,a}_{\lambda,\beta}(\mu), \\ 1 & \text{otherwise.} \end{cases}$$

For $\gamma = 1$, the proof is similar to the proof of 38 (using f^*_α, but now we get only $c^* \neq \emptyset$ mod $J^{\kappa,a}_\lambda$, which suffice. For $\gamma = 0$, take a close enough elementary submodel M of $(H(\chi), \in, <^*_\chi)$ of cardinality χ, choose $\beta(1) < \lambda$, $\beta(1) < \sup(M \cap \lambda)$ and then choose $\beta(0) \in \lambda \cap M$ such that for $\zeta < \xi$,

$$f_{\alpha_{\beta(0),\zeta}}(\theta) < \mathrm{Min}[M \cap (\theta + 1) - f_{\alpha_{\beta(1),\zeta}}(\theta)]$$

and is large enough.

47. Remark. (1) We can analyze $\Pi c / J^{\mathrm{bc}}_c$ when $\kappa = |c| < \min c$, $\sigma = \mathrm{cf}(c)$, $2^\sigma < \mathrm{Min}(c)$ define pcf, hence the $f_{\lambda,\alpha}$ etc. This essentially is what is used in proving the bound on $\aleph^{\mathrm{cf}\,\delta}_\delta$ when $\delta < \aleph_\delta$, $2^{\mathrm{cf}\,\delta} < \aleph_\delta$.

(2) A conclusion (of 25) is that for singular $\lambda > 2^{\mathrm{cf}\,\delta}$, there is a Kurepa tree for λ.

Note

48. Claim. If $\mathrm{Pr}_1(\lambda, \chi, \sigma)$ and $2^\chi \geqq \lambda$ then $\mathrm{Pr}_0(\lambda, \chi, \sigma)$.

Proof. As in [Sh 6] — 3.x (and explicitly in [Sh 11] 1.3).

48A. Remark. Note that our partition relations show that it is quite hard to get positive ones.

A slight strengthening of Todorcevic [T2] is

49. THEOREM. *If $\lambda \in R^{\kappa,2}_{\mathrm{ac}}$, κ minimal, then there are a $\lambda - L$-space and a $\lambda - S$-space with κ-directed nb basis for each point.*

PROOF. By the definition there are an $a \in I^{\kappa,2}$, $\sup a < \lambda$ $(2^\kappa < \lambda)$ and $f_\alpha \in \Pi a (\alpha < \lambda)$ such that for no $\alpha < \beta$, $f_\beta \leqq f_\alpha$. For repeating the Todorcevic proof [T2] we seemingly need also

$$(*) \qquad \text{for } \mu \in a, \qquad |\{f_\alpha \restriction \mu^+ : \alpha < \lambda\}| < \lambda.$$

We can get it w.l.o.g. by 22, 25 (we get really more).

In his proof a basis of clopen neighbourhoods of α is

$$u^S_{\alpha,\mu} = \{\beta < \lambda : f_\beta \leqq f_\alpha, f_\beta \restriction \mu^+ = f_\alpha \restriction \mu^+\}$$

for $\mu \in A$ (thus we get a $\lambda - S$-space).

For a $\lambda - L$-space use as a basis of clopen neighbourhoods of α

$$u^L_{\alpha,\mu} = \{\beta < \lambda : f_\alpha \leqq f_\beta, f_\beta \restriction \mu^+ = f_\alpha \restriction \mu^+\}.$$

§7. On $\lambda \to [\mu^+]^2_\theta$

50. LEMMA. *Suppose b is a set of regular cardinals $> 2^\kappa$, $|b| \leqq \kappa$, b has no last element, $\mathscr{D}$ is a filter on b to which all cobounded subsets of b belong,*

$$(\forall b')[b' \subseteq b \wedge |b'| < |b| \to b - b' \in \mathscr{D}]$$

and $\mu \geqq (\sup b)^+$, μ regular, $\theta \leqq \kappa$.

Suppose further $\lambda \geqq \mu$, $f_\alpha \in \Pi b$ for $\alpha < \lambda$ and $f_\alpha < f_\beta$ mod $\mathscr{D}$ for $\alpha < \beta < \lambda$.

If $a_i \subseteq b$ *for* $i < \theta$, $b - a_i \notin \mathscr{D}$, $[i \neq j \Rightarrow a_i \cap a_j = \varnothing]$ then *there is a symmetric two place function e from λ to θ such that*:

If $k < \omega$, $\alpha_{\beta,\zeta}$ *($\beta < \mu$, $\zeta < k$) are pairwise distinct ordinals $< \lambda$, $k < \omega$,* $h : \{1, \ldots, k\} \to \theta$, then *we can find $\beta(1) < \cdots < \beta(k) < \mu$ such that*:

(i) $\alpha_{\beta(1),\zeta} < \alpha_{\beta(2),\zeta} < \cdots < \alpha_{\beta(k),\zeta}$ *for* $\zeta < k$;

(ii) *for $\zeta(1) \neq \zeta(2)$ one of the following holds*:

  (a) *for* $l, m = 1, \ldots, k$, $\alpha_{\beta(l),\zeta(1)} < \alpha_{\beta(m),\zeta(2)}$;

  (b) *for* $l, m = 1, \ldots, k$, $\alpha_{\beta(l),\zeta(1)} > \alpha_{\beta(m),\zeta(2)}$;

  (c) *either*

  $\alpha_{\beta(1),\zeta(1)} < \alpha_{\beta(1),\zeta(2)} < \alpha_{\beta(2),\zeta(1)} < \alpha_{\beta(2),\zeta(2)} \cdots$

  or

  $\alpha_{\beta(1),\zeta(2)} < \alpha_{\beta(1),\zeta(1)} < \alpha_{\beta(2),\zeta(2)} < \alpha_{\beta(2),\zeta(2)} \cdots$

(iii) *for* $l < m$, $\zeta < k$: $e(\alpha_{\beta(l),\zeta}, \alpha_{\beta(m),\zeta}) = h(l)$;

(iv) *for* $\zeta(1) \neq \zeta(2) < k$,

(a) *for* $l \neq m$, $e(\alpha_{\beta(l),\zeta(1)}, \alpha_{\beta(m,\zeta(2))}) = 0$

or

(b) *for* $l < m$, $e(\alpha_{\beta(l),\zeta(1)}, \alpha_{\beta(m,\zeta(2))}) = h(l)$.

50A. REMARK. In (iv) there are some restrictions on $\{(\zeta(1), \zeta(2)) : \text{(iv)(a)}$ occurs$\}$; see the proof.

PROOF. We define the colouring as follows: θ, e are symmetric functions; for $\alpha < \beta$

$$\theta(\alpha, \beta) = \mathrm{Max}\{\theta \in b : f_\alpha(\theta) > f_\beta(\theta)\}$$

$$(\theta(\alpha, \beta) \text{ is undefined if there is no maximal such as } \theta),$$

$$e(\alpha, \beta) = \begin{cases} i & \text{if } \theta(\alpha, \beta) \in a_i, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $k < \omega$, $\langle \alpha_{\beta,\zeta} : \zeta < k \rangle$ is a sequence of ordinals $< \lambda$ for $\beta < \mu$ such that:

$$\alpha_{\beta(1),\zeta(1)} = \alpha_{\beta(2),\zeta(2)} \Leftrightarrow \beta(1) = \beta(2) \wedge \zeta(1) = \zeta(2).$$

Let $\alpha(\beta, \zeta) = \alpha_{\beta,\zeta}$.

Let χ be regular large enough, $<^*_\chi$ a well ordering of $H(\chi)$.

Let $\langle N_i : i < \kappa^+ \rangle$ be increasing continuous, $\| N_i \| = 2^\kappa$, $N_i < (H(\chi), \in, <^*_\chi)$, $\langle N_j : j \leqq i \rangle \in N_{i+1}$ and every subset of N_{i+1} of cardinality $\leqq \kappa$ belongs to N_{i+1} and b, $\langle f_\alpha : \alpha < \lambda \rangle$, $\langle \langle \alpha_{\beta,\zeta} : \zeta < k \rangle : \beta < \mu \rangle$ belongs to N_0. Let $N_{\kappa^+} = \bigcup_{i < \kappa^+} N_i$.

Choose $\beta(*) < \mu$, $\beta(*) > \sup(\mu \cap \bigcup_{i<\kappa^+} N_i)$. For each $\zeta < k$ let $b_\zeta = \{\sigma \in b : f_{\alpha(\beta(*),\zeta)}(\sigma) \in \bigcup_{i<\kappa^+} N_i\}$, $b^\zeta = b - b_\zeta$, and f^*_ζ, f^+_ζ be defined by:

$$\mathrm{dom}\, f^*_\zeta = b^\zeta,$$

$$f^*_\zeta(\sigma) = \mathrm{Min}\{j : j \in \bigcup_{i<\kappa^+} N_i, j > f_{\alpha(\beta(*),\zeta)}(\sigma)\},$$

$$\mathrm{dom}\, f^+_\zeta = b,$$

$$f^+_\zeta = f^*_\zeta \cup (f_{\alpha(\beta(*),\zeta)} \restriction b_\zeta).$$

Clearly for some successor $i(*) < \kappa^+$, $\mathrm{rang}\, f^+_\zeta \subseteq N_{i(*)}$ for every $\zeta < k$, hence $\langle f^+_\zeta : \zeta < k \rangle \in N_{i(*)}$.

Let, for $\zeta(1) \neq \zeta(2) < k$, $c_{\zeta(1),\zeta(2)} = \{\sigma \in b : f_{\alpha(\beta(*),\zeta(1))}(\sigma) = f_{\alpha(\beta(*),\zeta(2))}(\sigma)\}$. For $\sigma \in b$ let

$$N^\sigma_{i+1} = \text{Skolem Hull in } (H(\chi), \in, <^*_\chi) \text{ of } N_{i+1} \cup \{f_{\alpha(\beta(*),\zeta)}(\sigma) : \zeta < k\},$$

$$N^\sigma_\delta = \bigcup_{i<\delta} N^\sigma_{i+1}.$$

We can find $\beta(i,\sigma) \in \mu \cap N^\sigma_{i(*)+i+2}$ such that

(a) $f_{\alpha(\beta(i,\sigma),\zeta)}(\sigma_1) = f_{\alpha(\beta(*),\zeta)}(\sigma)$ for $\zeta < k$,

(b) $f_{\alpha(\beta(i,\sigma),\zeta)}(\tau) = f_{\alpha(\beta(*),\zeta)}(\tau)$ for $\zeta < k$, $\tau \in b_\zeta$,

(c) $\sup(f^*_\zeta(\tau) \cap N_{i(*)+i+1}) < f_{\alpha(\beta(i,\sigma),\zeta)}(\tau) < \sup f^*_\zeta(\tau) \cap N_{i(*)+i+2}$ for $\zeta < k$, $\tau \in b - b_\zeta - \sigma^+$.

(d) $f_{\alpha(\beta(*),\zeta_1)}(\theta_1) < f_{\alpha(\beta(*),\zeta_2)}(\theta_2)$ *iff* $f_{\alpha(\beta(i,\sigma),\zeta_1)}(\theta_1) < f_{\alpha(\beta(i,\sigma),\zeta_2)}(\theta_2)$ for any $\zeta_1, \zeta_2 < \xi$ and $\theta_1, \theta_2 \in b$.

Now suppose $i_1 < i_2 < i(*)$, and $\zeta_1, \zeta_2 < k$ and $\sigma_2 < \sigma_1 \in b - \bigcup_\zeta b_\zeta$ and we shall check the color of $\alpha_{\beta(i_1,\sigma_2),\zeta_1}, \alpha_{\beta(i_2,\sigma_2),\zeta_2}$.

Case I: $\zeta_1 = \zeta_2 = \zeta$.

Easily

$$b_\zeta \subseteq \{\tau : f_{\alpha(\beta(i_1,\sigma_1),\zeta)}(\tau) = f_{\alpha(\beta(i_2,\sigma_2),\zeta)}(\tau)\} \subseteq b_\zeta \cup (b \cap \sigma_2^+),$$

$$\{\sigma_2\} \subseteq \{\tau : f_{\alpha(\beta(i_1,\sigma_1),\zeta)}(\tau) > f_{\alpha(\beta(i_2,\sigma_2),\zeta)}(\tau)\} \subseteq \{\sigma_2\} \cup (b \cap \sigma_2).$$

So $e(\alpha_{\beta(i_1,\sigma_1),\zeta_1}, \alpha_{\beta(i_2,\sigma_2),\zeta_2}) = h(\sigma_2)$.

Case II: $\zeta_1 \neq \zeta_2$.

$$\begin{aligned} b_{\zeta_1} \cap b_{\zeta_2} \cap c_{\zeta_1,\zeta_2} &\subseteq \{\tau \in b : f_{\alpha(\beta(i_1,\sigma_1),\zeta_1)}(\tau) = f_{\alpha(\beta(i_2,\sigma_2),\zeta_2)}(\tau)\} \\ &\subseteq (b_{\zeta_1} \cap b_{\zeta_2} \cap c_{\zeta_1,\zeta_2}) \cup (b \cap \sigma_2), \end{aligned}$$

$$\begin{aligned} \{\tau \in b : f^+_{\zeta_1}(\tau) > f^+_{\zeta_2}(\tau)\} \cup \{\sigma_2\} &\subseteq \{\tau : f_{\alpha(\beta(i_1,\sigma_1),\zeta_1)}(\tau) > f_{\alpha(\beta(i_2,\sigma_2),\zeta_2)}(\tau)\} \\ &\subseteq \{\tau \in b : f^+_{\zeta_1}(\tau) > f^+_{\zeta_2}(\tau)\} \cup \{\sigma_2\} \cup (b \cap \sigma_2). \end{aligned}$$

Now choose $i(*) < i_1 < \cdots i_k < \kappa^+$, $\sigma_1 > \cdots > \sigma_k$ in $b - \bigcup_\zeta b_\zeta$, such that if $\{\sigma : f^+_{\zeta(1)}(\sigma) > f^+_{\zeta(2)}(\sigma)\}$ is bounded in b then σ_k is a bound. Now $\{\alpha_{\beta(i_l,\sigma_l),\zeta} : l,\zeta\}$ are as required.

51. Lemma. *Suppose $b, \kappa, \mathscr{D}, \mu = (\sup b)^+, \lambda, f_\alpha$ $(\alpha < \lambda)$ are as in Lemma 50, and suppose further that for $\sigma \in b$, $\{f_\alpha \restriction (b \cap \sigma) : \alpha < \lambda\}$ has cardinality $< \sigma$ (see 22, 25 for such cases), and $\xi < \kappa^+$, $(\forall \sigma \in b)(\forall \theta < \sigma)[\theta^{|\xi|} < \sigma]$.*

(1) *For every $A \subseteq \lambda$, $|A| = \mu$, there is $b_1 \in \mathscr{D}$ such that:*

for every $k < \omega$, $m < \omega$ and one to one functions $h_l : \{0, \ldots, k-1\} \to \omega$ and

Min $b = \sigma_0 < \sigma_2 < \cdots < \sigma_m$ *from* b_1 there are $\alpha_0 < \alpha_1 < \cdots < \alpha_{k-1}$ *from* A *such that for* $i < j < k$: $p \leqq m$

$$(\forall \sigma \in b_1)[f_{\alpha_i}(\sigma) = f_{\alpha_j}(\sigma)],$$

$$(\forall \sigma)[\sigma \in b \wedge \sigma \notin b_1 \wedge \sigma \in [\sigma_p, \sigma_{p+1}) \rightarrow f_{\alpha_i}(\sigma) < f_{\alpha_j}(\sigma) \equiv h_p(i) \leqq b_p(j)],$$

$$(\forall \sigma)[\sigma \in b \wedge \sigma \notin b_1 \wedge \sigma \geqq \sigma_m \rightarrow f_{\alpha_i}(\sigma) < f_{\alpha_j}(\sigma)].$$

(2) *For every* $\xi < \kappa^+$, $\alpha_{\beta,\zeta}$ $(\beta < \mu, \zeta < \xi)$ *distinct ordinals* $< \lambda$, $r(*) < \kappa^+$, *the following holds*:

For some $b_\zeta \in \mathscr{D}$, $f_\zeta^+ : b \rightarrow \mathrm{ord}$ (*for* $\zeta < \xi$):

For every *sequence* $\langle k_r \langle b_r^0, b_r^1 : r < r \rangle \rangle$ *such that* $k_n < \omega$, b_r^0 *is a strictly increasing function from* $k_r = \{0, \ldots, k_i - 1\}$ *to ordinal* $[r \neq s \Rightarrow$ $\mathrm{rang}(b_r^0) \cap \mathrm{rang}(b_s^0) = \emptyset]$, b_r' *a strictly decreasing function from* $k_r = \{0, \ldots, k_r - 1\}$ *to* b there are $\beta(r)(r < r(*))$, *from* A, *such that*:

(a) $r < j < i(0) \Rightarrow p(r) < \beta(s)$.

(b) *Let* $\zeta(1), \zeta(2) < \xi$, $i < j < i(*)$,

$$\{\sigma \in b : f_{\alpha(\beta(r),\zeta(1)}(\sigma) = f_{\alpha(\beta(s),\zeta(2)}(\sigma)\}$$
$$= b_{\zeta(1)} \cap b_{\zeta(2)} \cap \{\sigma \in b : f_{\zeta(1)}^+(\sigma) = f_{\zeta(2)}^+(\sigma)\}.$$

(c) *Let* $\zeta(1), \zeta(2) < \xi$, $r < s$, $\sigma \in b$, *if* $f_{\zeta(1)}^+(\sigma) < f_{\zeta(2)}^+(\sigma)$ *then* $f_{\alpha(\beta(r),\zeta(1))}(\sigma) < f_{\alpha(\beta(s),\zeta(2))}(\sigma)$.

(d) *Let* $\zeta(1), \zeta(2) < \xi$, $r < s$, $\sigma \in b$, $f_{\zeta(1)}^+(\sigma) = f_{\zeta(2)}^+(\sigma)$; *if* $\sigma \in b_{\zeta(2)} - b_{\zeta(1)}$ *then* $f_{\alpha(\beta(r),\zeta(1))}(\sigma) < f_{\alpha(\beta(s),\zeta(1))}(\sigma)$ *while if* $\sigma \in b_{\zeta(1)} - b_{\zeta(2)}$ *then* $f_{\alpha(\beta(r),\zeta(1))}(\sigma) > f_{\alpha(\beta(s),\zeta(2))}(\sigma)$.

(e) *Let* $\zeta(1), \zeta(2) < \xi$, $r < s$, $\sigma \in b - b_{\zeta(1)} \cup b_{\zeta(2)}$, *and* $f_{\zeta(1)}^+(\sigma) = f_{\zeta(2)}^+(\sigma)$.

If $l \leqq k_r m \leqq k_j$, $h_r^1(l) \leqq \sigma < h_r^1(l+1)$, $h_s^1(m) \leqq \sigma < h_s^1(m+1)$ (*stipulating* $h_r^1(k_1) = \sup b$), *then*

$$f_{\alpha(\beta(r),\zeta(1))}(\sigma) < f_{\alpha(\beta(s),\zeta(2))}(\sigma) \Leftrightarrow h_r^0(l) \leqq h_s^0(m).$$

51A. Remark. The proof gives more than stated in the lemma.

Proof. (1) Follows by 2.

(2) Let $\alpha_{\beta,\zeta}$ $(\beta < \mu, \zeta < \xi)$ be distinct ordinals $< \lambda$ (for some $\xi < \kappa^+$). As in the proof of Lemma 50 we choose χ, $<_\chi^*$, $\langle N_i : i < \kappa^+ \rangle$, and then $\beta(*) < \mu$, b_ζ, $g\, f_\zeta^*$, f_ζ^+ for $\zeta < \xi$ and $i(*)$. Note

(*) if $\sigma \in b$, $\alpha_\zeta < \lambda$ for $\zeta < \xi$, $\bar{f} = \langle f_{\alpha_\zeta} \upharpoonright \sigma : \zeta < \xi \rangle$;
$N_{r+1}[\bar{f}]$ is the Skolem Hull of $|N_{i+1}|$ $\{\bar{f}\}$, and
$\theta \in b - \bigcup_l \sigma^+$ *then* $\sup(\theta \cap N_{i+1}[\bar{f}]) = \sup(\theta \cap N_{i+1})$.

We shall show that b_ζ, f_ζ^+ ($\zeta < \xi$) are as required, so let $\langle k_r, b_r^0, b_r^1 : r < r(*)\rangle$.

W.l.o.g. each b_r^0 is into $\{i+1 : i(*) < i < \kappa^+\}$. Now for each $r < r(*)$, we define $\beta(r, l)$, $l = 0, \ldots, k_r$ by downward induction on l. For $l = k_r$, $\beta(r, l) = \beta(*)$. If $\beta(r, l+1)$ is defined, *let*: $N_{r,l}$ be Skolem Hull of

$$N_{h_r^0(l)}[\langle f_{\alpha(\beta(r,l+1),\zeta)} \restriction h_r^0(l) : \zeta < \xi\rangle]$$

and *let*: $\beta(r, l) \in N_{r,l} \cap \mu$ be such that

$$\langle f_{\alpha(\beta(r,l),\zeta)} \restriction h_r^0(l) : \zeta < \xi\rangle = \langle f_{\alpha(\beta(r,l+1),\zeta)} \restriction h_r^0(l) : \zeta < \xi\rangle,$$

$$\bigwedge_{\zeta<\xi} (\forall \sigma \in b_\zeta) f_{\alpha(\beta(r,l),\zeta)}(\sigma) = f_{\alpha(\beta(r,l+1),\zeta)}(\sigma) = f_{\alpha(\beta(*),\zeta)},$$

$$\bigwedge_{\zeta<\xi} (\forall \sigma \in b - b_\zeta)[f_{\alpha(\beta(r,l),\zeta)}(\sigma) \notin N_{h_r^0(l)-1}],$$

$$\bigwedge_{\zeta<\xi} (\forall \sigma \in b - b_\zeta - h_r^0(l))[\forall \gamma \in o \cap N_{h_r^0(l)}(f_{\alpha(\beta(r,l),\zeta)}(\sigma) \leqq \gamma \equiv f_\zeta^+(\sigma) \leqq \gamma)].$$

There is no problem, and this suffices.

52. Conclusion. Under the assumptions of Lemma 51, $\lambda \to [\mu]_\theta^2$.

53. Conclusion. If, in Lemma 51, $\theta > k$, $k < \omega$, $\mu = (\sup b)^+$ *then* there are Boolean algebras $B_1, \ldots, B_k$ such that

(i) each B_l satisfies the μ^+-c.c., and even

$$\prod_{\substack{i=1,k \\ i \neq l}} B_i \text{ satisfies the } \mu^+\text{-c.c.}$$

(ii) $\Pi_{i=1}^k B_i$ does not satisfy the λ^+-c.c.

Proof. Let e be a two place function from λ to θ as there. For $l = 1, \ldots, k$ let B_l be the Boolean algebra generated freely by $\{x_\beta : \beta < \lambda\}$ except that $x_\beta \cap x_\gamma = 0$ when $e(\beta, \gamma) = l$. Also the rest is as in [Sh 7] 1.10.

§8. $\aleph_\omega^{\aleph_0} < \aleph_{(2^{\aleph_0})^+}$ revisited

We can generalize the information on Πa to more sets.

54. Definition. (1) $I^{\kappa,3} = \{A : A \subseteq R^{\kappa,2}, \operatorname{Min} A > |A|^\kappa\}$ (κ will be fixed below).

(2) $\operatorname{pcf}^\kappa(A) = \bigcup\{\operatorname{pcf}(b) : b \subseteq a, |b| \leqq \kappa\} = \{\operatorname{cf}(\Pi A, <_{\mathscr{D}}) : \mathscr{D} \in uf^\kappa(A)\}$ where

$$fl^\kappa(A)=\{\mathscr{D}:\mathscr{D}\text{ a filter on }A,\ \mathrm{Min}\{|b|:b\in\mathscr{D}\}\leqq\kappa\},$$

$$uf^\kappa(A)=\{\mathscr{D}\in fl^\kappa(A):\mathscr{D}\text{ an ultrafilter on }A\}.$$

(3) For $f,g\in\Pi A$, $A\in I^{\kappa,3}$, $f\leqq_{\lambda,\kappa}g$ *iff* for every $b\in[A]^\kappa$,

$$[b\in J^{\kappa,b}_{<\lambda^+}\Rightarrow f\restriction b\leqq_{J^{\kappa,b}_{<\lambda}}g\restriction b].$$

55. Claim. (1) For $A\in I^{\kappa,3}$, $|\mathrm{pcf}^\kappa(A)|\leqq|A|^\kappa+2^\kappa$.

(2) For $A\in I^{\kappa,3}$, $\lambda\in\mathrm{pcf}(A)$, we can define $\langle f^{\kappa,A}_{\lambda,\alpha}:\alpha<\lambda\rangle$ such that:

(i) $\alpha<\beta\Rightarrow f^{\kappa,A}_{\lambda,\alpha}<_{\lambda,\kappa}f^{\kappa,A}_{\lambda,\beta}$,

(ii) for every $g\in\Pi A$ for some $\alpha<\lambda$

$$g<_{\lambda,\kappa}f^{\kappa,A}_{\lambda,\alpha}.$$

(3) For $A\in I^{\kappa,3}$, $\lambda\in\mathrm{pcf}^\kappa(A)$ there is $B=B^A_\lambda$ such that

(i) $B\subseteq A$,

(ii) for every $a\in[A]^{\leqq\kappa}$

$$a\in J^{\kappa,a}_{<\lambda^+}\Leftrightarrow a-B\subset J^{\kappa,a\cap B}_{\leqq\lambda}.$$

(4) For $A\in I^{\kappa,3}$, and λ and $a\in[A]^{\leqq\kappa}$: $\max\mathrm{pcf}(a)<\lambda\Leftrightarrow a$ is in the ideal of $\mathscr{P}(A)$ generated by $\{B^A_\mu:\mu\in\lambda\cap\mathrm{pcf}^\kappa(A)\}$.

(5) If $A=\{\theta:\aleph_\alpha\leqq\theta<\aleph_\delta,\ \theta\text{ regular}\}$, $\aleph^\kappa_\alpha<\aleph_\delta$, then $\mathrm{pcf}^\kappa(A)$ has the form $\{\theta:\aleph_\alpha\leq\theta<\aleph_\gamma,\ \theta\text{ regular}\}$ for some γ.

Proof. (1) Easy.

(2) We define by induction on $\alpha<\lambda$, a function $f^*_\alpha=f^{\kappa,A}_{\lambda,\alpha}\in\Pi A$ and $\langle\gamma^a_\alpha:a\in L\rangle$ where

$$L=\{a:a\in[A]^{\leqq\kappa},\ a\in J^{\kappa,a}_{<\lambda^+}-J^{\kappa,a}_{<\lambda}\}$$

such that

(a) $f^{\kappa,a}_{\lambda,\gamma^a_\alpha}\restriction a\leqq f^*_\alpha\restriction a$ for $a\in L$,

(b) $\alpha\leqq\gamma^a_\alpha<\lambda$ for $a\in L$,

(c) for $\beta<\alpha$, $a\in L$, $f^*_\beta\restriction a<f^{\kappa,a}_{\lambda,\gamma^a_\alpha}$ mod $J^{\kappa,a}_{<\lambda}$.

There is no problem to do it [for each α, first choose $\langle\gamma^a_\alpha:a\in L\rangle$ such that (b) and (c) are satisfied, next choose f^*_α such that (a) is satisfied; this is possible as $\mathrm{Min}\,A>|A|^\kappa\geqq|L|$, each $\theta\in A$ regular]. Now $\langle f^*_\alpha:\alpha<\lambda\rangle$ is as required.

(3) Let $\langle f^{\kappa,A}_{\lambda,\alpha}:\alpha<\lambda\rangle$ be from (2). For every $a\in L_1=\{a\in[A]^{\leqq\kappa}:a\notin J^{\kappa,a}_{<\lambda^+}\}$, we can find $h_a\in\Pi a$ such that, for every $\alpha<\lambda$, $h_a\nleqq f^{\kappa,A}_{\lambda,\alpha}\restriction a$ mod $J^{\kappa,a}_{<\lambda}$.

We can define $h\in\Pi A$,

$$h(\theta)=\sup[\{0\}\cup\{h_a(\theta):\theta\in a\in L_1\}].$$

Now for every $b \in L$ for some $\gamma_b < \lambda$, $h \restriction b \leqq f^{\kappa,b}_{\lambda,\gamma_b} \bmod J^{\kappa,b}_{<\lambda}$. Let $\gamma(*) = \sup_{b \in L} \gamma_b < \lambda$, so for $b \in L$, $h \restriction b \leqq_{\lambda,\kappa} f^{\kappa,A}_{\lambda,\gamma(*)}$. Let $B = \{\theta \in A : h(\theta) \leqq f^{\kappa,A}_{\lambda,\gamma(*)}(\theta)\}$.

(4) Easy.

(5) By 5(2).

56. Remark. By [Sh 5] $(\aleph_\delta)^{\mathrm{cf}\,\delta} < \aleph_{(|\delta|^{\mathrm{cf}\,\delta})^+}$ and when $\aleph_\alpha^{\mathrm{cf}(\delta)} < \aleph_{\alpha+\delta}$, $(\aleph_{\alpha+\delta})^{\mathrm{cf}\,\delta} < \aleph_{\alpha+(|\delta|^{\mathrm{cf}\,\delta})^+}$.

57. The simpler proof of $(\aleph_{\alpha+\delta})^{|\delta|} < \aleph_{\alpha+(2^{|\delta|})^+}$ when $\aleph_\alpha^{\mathrm{cf}\,\delta} < \aleph_{\alpha+\delta}$.

Let $\kappa = |\delta|$, and δ a minimal counterexample. The case $\delta = \aleph_{\alpha+\delta}$ is unreal so let $\delta < \aleph_{\alpha+\delta}$. For some α_1, δ_1, $\delta = \alpha_1 + \delta_1$, $|\delta_1| < \aleph_{\alpha_1+\delta_1}$, α_1 limit or zero and $|\delta_1|^\kappa \leqq \aleph_{\alpha_1}$ or $\delta_1 = \omega$. If $(\aleph_{\alpha+\alpha_1})^\kappa \geqq \aleph_{\alpha+\delta}$ we can finish easily. So w.l.o.g. (remember $\aleph^\kappa_{\alpha+1} \leqq \aleph^\kappa_\alpha + \aleph_{\alpha+1}$) $|\delta|^\kappa \leqq \aleph^\kappa_\alpha$. Let $\mu = (\aleph_\alpha)^\kappa$, $a = \{\lambda : \mu^+ < \lambda < \aleph_\delta, \lambda \text{ regular}\}$. So $|a| \leqq |\delta| < \aleph_\alpha \leqq \mu < \mathrm{Min}\, a$, so 16 applies.

Let $\langle f^{a,\kappa}_{\lambda,\alpha} : \alpha < \lambda \rangle$ be as in Claim 55.

Let χ be regular large enough, $<^*_\chi$ a well ordering of $H(\chi)$. For each $x \subseteq \aleph_{\alpha+\delta}$, $|x| \leqq \kappa$ we define by induction on $\zeta < \mu^+$, N^x_ζ such that

(a) $N^x_\zeta < (H(\chi), \in, <^*_\zeta)$,

(b) $\| N^x_\zeta \| = \mu$, $\mu + 1 \subseteq N^x_\zeta$, $x \in N^x_\zeta$, $\langle f^{\kappa,a}_{\lambda,\alpha} : < \chi \rangle : \lambda \in \mathrm{pcf}\, a \in N^x_\zeta$, $A \in N^x_\zeta$,

(c) $\langle N^x_\zeta : \zeta < \mu^+ \rangle$ is increasing continuous,

(d) $\langle N^x_\zeta : \zeta \leqq \xi \rangle \in N^x_{\xi+1}$.

As $\mu + 1 \subseteq N^x_\zeta$, $\mathrm{pcf}(A) \in N^x_\zeta$ clearly $\mathrm{pcf}(a) \subseteq N^x_\zeta$. Let ch^x_ζ is the function with domain a, $ch^x_\zeta(\theta) = \sup(\theta \cap N^x_\zeta)$, it is $< \theta$ as $|N^x_\zeta| < \mu^+ = \mathrm{Min}\, A$ hence $ch^x_\zeta \in \Pi a$. So for each $\lambda \in \mathrm{pcf}(a)$ and ζ, for some $\alpha_\lambda(x, \zeta) < \lambda$ (letting $b = b^{a,\kappa}_\lambda$)

$$f^{b,\kappa}_{\lambda,\alpha(x,\zeta)} < ch^x_\zeta \restriction b \leqq f^{b,\kappa}_{\lambda,\alpha_\lambda(x,\zeta)} \bmod J^{\kappa,b}_{<\lambda}.$$

Let

$$N^x_{\mu^+} = \bigcup_{\zeta < \mu^+} N^x_\zeta, \qquad ch^x_{\mu^+}(\theta) = \sup_\zeta\, ch^x_\zeta(\theta) = \sup(N^x_{\mu^+} \cap \theta).$$

Let $\alpha_\lambda(x) = \bigcup_\zeta \alpha_\lambda(x, \zeta)$. So easily $ch^x_{\mu^+} \restriction b = f^{b,\kappa}_{\lambda,\alpha_\lambda(x)} \bmod J^{\kappa,b}_{<\lambda}$. So

$$ch^x_{\mu^+} = \bigcup_{l=1}^{h} f^{b_l,\kappa}_{\lambda_l,\alpha_{\lambda_l}(x)} \restriction b_l \cap c_l$$

where $\langle b_l, c_l : l = 1, \kappa \rangle$ depend on x. So the number of possible $ch^x_{\mu^+}$ is $\leqq \sup \mathrm{pcf}\, a$ which, by 55(5), is $\leqq \aleph_{\alpha+(2^{|\delta|})^+}$. Now $ch^x_{\mu^+}$ determines $N^x_{\mu^+} \cap \aleph_{\alpha+\delta}$ as

(*) if $M_1, M_2 \prec (H(\chi), \in, <^*_\chi)$, $\mu^+ \subseteq M_l$, $\delta \in M_l$, for each $\theta \in A$, $\sup(M_1 \cap \theta) = \sup(M_2 \cap \theta)$, it has cofinality $> \aleph_0$, and both M_1 and M_2 contains a club for it *then* $M_1 \cap \aleph_\delta = M_2 \cap \aleph_\delta$, so we finish.

58. THEOREM. *Suppose* $\mathrm{cf}(\delta) = \kappa$, $(\forall i < \delta)(\aleph_i^\kappa < \aleph_\delta)$,

$$\mu \stackrel{\text{def}}{=} |\aleph_\delta \cap R^\kappa_{\text{ia}}| < \aleph_\delta \qquad \text{then } \aleph_\delta^\kappa < \aleph_{\delta+(\mu^\kappa)^+}.$$

59. REMARK. If $\aleph_\delta$ is minimal such that $\aleph_\delta = \delta$, $\kappa = \aleph_0$ then $\aleph_\delta \cap R^\kappa_{\text{ia}} - (2^{\aleph_0})^+ = \{\theta : \theta = \mathrm{cf}\,\theta = \theta^{\aleph_0}, 2^{\aleph_0} < \theta < \aleph_\delta\}$.

60. PROOF. Either imitate proof of the bound or show inductively that $|\aleph_\delta \cap R^\kappa_{\text{ia}}|^\kappa \leqq |\delta|^\kappa$.

61. THEOREM. *If* $S = \{\delta < \aleph_2 : \mathrm{cf}\,\delta = \aleph_0, \aleph_\delta^{\aleph_0} < \aleph_{\delta+\omega_1}\}$ *is stationary*, $2^{\aleph_1} < \aleph_{\omega_1}$ *then* $\aleph_{\omega_1}^{\aleph_1} < \aleph_{\omega_2}$.

PROOF. By 63.

62. THEOREM. *Suppose* $\mathrm{cf}(\delta) \leqq \theta < \kappa$, *and for every* $a \subseteq R^{\kappa,2} \cap \aleph_\delta$,

$$[\,|a| \leqq \theta \Rightarrow \max \mathrm{pcf}(a) \leqq \aleph_{\delta+\alpha(*)}]$$

where $\alpha(*) < \theta^+$, *and* $2^\kappa < \aleph_\delta$.
Then for some $\gamma < \delta$, *every* $a \subseteq R^{\kappa,2} \cap [\aleph_\gamma, \aleph_\delta)$,

$$[\,|a| \leqq \kappa \Rightarrow \max \mathrm{pcf}(a) \leqq \aleph_{\delta+\alpha(*)}].$$

62A. REMARK. If $(\aleph_\delta)^\theta < \aleph_{(\delta+\theta^+)}$, the assumption holds.[6]

PROOF. Now let $A \subseteq R^\kappa \cap [(2^\kappa)^{++}, \aleph_\delta]$ be a set of cardinality $\leqq \kappa$, $\langle f^{A,\theta}_{\lambda,\beta} : \beta < \lambda \rangle \lambda \in c \stackrel{\text{def}}{=} \{\aleph_{\delta+\beta+1} : \beta < \alpha(*)\}$ are well defined. Now if $f \in \Pi A$ for each λ for some $\beta(f,\lambda) < \lambda$, $f <_{\lambda,\theta} f^{A,\theta}_{\lambda,\beta(f,\lambda)}$. If for every finite $w \subseteq c$, $f \not\leqq \mathrm{Max}_{\lambda \in w} f^{A,\theta}_{\lambda,\beta(f,\lambda)}$ then there is $\sigma(w) \in A$ which exemplifies it. Now let

$$b = \bigcup \{\sigma(w) : w \subseteq c;\ w \text{ finite}\}.$$

For $f \restriction b$ we can find a w, hence we have shown $\mathrm{cf}\,\Pi A \leqq \aleph_{\delta+\alpha(\lambda)}$; this shows the desired conclusion.

63. THEOREM. *Suppose* $\mathrm{cf}\,\kappa = \kappa > \theta$, $\mu^{<\kappa} < \aleph_\mu$, $\mathrm{cf}\,\mu = \mu$ *and*

$$S_1 = \{\delta < \mu : (\aleph_\delta)^\theta < \aleph_{(\delta+\theta^+)}, \mathrm{cf}(\delta) = \theta\}$$

or

[6] We can, in the assumption, restrict ourselves to $[\aleph_\alpha, \aleph_\delta) \supseteq a$ for some fixed $\alpha < \delta$.

$$S_2 = \{\delta < \mu : \operatorname{cf} \delta = \theta \text{ and for some } \alpha < \delta \text{ and } \gamma < \mu, \text{ for every } a \in R^{\theta,2},$$
$$a \subseteq [\aleph_\alpha, \aleph_\delta] \Rightarrow \max \operatorname{pcf} a < \aleph_\gamma\}$$

is stationary.

Then *for every* $\alpha < \mu$, $(\aleph_\alpha)^{<\kappa} < \aleph_\mu$.

64(1). REMARK. In the definition of S we need just the conclusion of the last theorem.

PROOF. Let $\alpha(*)$ be minimal such that $(\aleph_{\alpha(*)})^{<\kappa} \geqq \aleph_\mu$ (assuming that the theorem fails). So $\aleph_{\alpha(*)} > \mu^{<\kappa}$, hence $\alpha(*)$ is limit, $[\alpha < \alpha(*) \Rightarrow \aleph_\alpha^{<\kappa} < \aleph_\mu]$; but $(\chi^{<\kappa})^{<\kappa} = \chi^{<\kappa}$ (as κ is regular) so $[\alpha < \alpha(*) \Rightarrow \aleph_\alpha^{<\kappa} < \aleph_{\alpha(*)}]$. So necessarily $\operatorname{cf}(\alpha(*)) < \kappa$, and $(\aleph_{\alpha(*)})^{\operatorname{cf}(\alpha(*))} \geqq \aleph_\mu$ so $\kappa > \sigma \overset{\text{def}}{=} \operatorname{cf}(\alpha(*))$. By the last theorem and Fodor we know that for some $\delta(*) < \mu$, $> \alpha(*)$, S is stationary where

$$S = \{\delta < \mu : \operatorname{cf}(\delta) = \theta, \aleph_\delta > \mu^\sigma, \text{ and } \sup \operatorname{pcf}^\sigma\{\aleph_{\gamma+1} : \delta(*) \leqq \gamma < \delta\} < \aleph_\mu\}.$$

Let for $\delta \leqq \mu$, $A(\delta) = \{\aleph_\alpha : \mu^\sigma + \aleph_{\delta(*)} < \aleph_\alpha < \aleph_\delta, \aleph_\alpha \text{ regular}\}$. Let χ be regular large enough, $N_\zeta \prec (H(\chi), \in, <^*_\chi)$ for $\zeta < \mu_0^+$, $\mu_0 \overset{\text{def}}{=} \mu^\sigma + \aleph_{\delta(*)}$, $\mu_0^+ \subseteq N_\zeta$, $\langle N_\xi : \xi \leqq \zeta \rangle \in N_{\zeta+1}$, and $\langle\langle\langle f_{\lambda,\alpha}^{A(\delta),\sigma} : \alpha < \lambda \rangle : \lambda \in \operatorname{pcf}^\sigma A(\delta)\rangle : \delta \in S\rangle$ belongs to N_0, as well as letting $B = \{\aleph_\alpha : \aleph_\alpha \text{ regular} < \aleph_{\alpha(*)}, \aleph_\alpha > \mu + 2^\sigma\}$, $\langle\langle f_{\lambda,\alpha}^{B,\kappa} : \alpha < \lambda \rangle : \lambda \in \operatorname{pcf} B\rangle$ and $\| N_\zeta \| = \mu_0^+$.

Now (if $\sigma > \aleph_0$ by 12 (holds if S_1 is stationary) and if $\sigma = \aleph_0$ by [Sh 5] Ch XIII (or see 65 below)) we know that, for every $\beta < \alpha(*)$ for some $a_\beta \in R^{\sigma,2}$, $a_\beta \subseteq (\mu^\sigma, \aleph_{\alpha(*)})$ and $\aleph_{\beta+1} \in \operatorname{pcf}(a_\beta)$. Let $W = \{\beta < \mu : \beta \geqq \delta(*)\}$.

Let g be the function: $g(\theta) = \sup(\theta \cap \bigcup_\zeta N_\zeta)$ for $\theta \in [\mu_0^{++}, \aleph_\mu)$. For each $\beta \in W$ let $b_\beta \subseteq B$ be maximal s.t.

$$g \restriction b_\beta = f^{a,\kappa}_{\aleph_{\beta+1}, g(\aleph_{\beta+1})} \restriction b_\beta.$$

Now there is a set $w_0 \subseteq W$ of cardinality $\leqq |B| \leqq |\alpha(*)| < \mu$ such that

$$\bigcup\{b_\beta : \beta \in w_0\} = \bigcup\{b_\beta : \beta \in W\}.$$

Choose $\gamma \in S$, $\gamma > \delta(*)$, $\gamma > \sup(w_0)$ and (note: C is a club of μ as $S \in \mu$ is stationary)

$$\gamma \in C \overset{\text{def}}{=} \{\delta < \mu : \text{if } a \subseteq R \cap [\aleph_{\delta(*)}, \aleph_\delta), |a| \leqq \sigma, \sup a < \aleph_\delta, \text{ then}$$
$$\sup \operatorname{pcf}(a) < \aleph_\delta\}.$$

So there is $w \subseteq w_0$, $|w| \leqq \sigma$, s.t. $b_\gamma \subseteq \bigcup_{\beta \in w} b_\beta$.

Let $\delta = \sup w$, $a = \{\aleph_{\beta+1} : \beta \in w\}$, $a^* = \mathrm{pcf}(a)$. Note that $a \subseteq a^* \subsetneq [\aleph_{\delta(*)}, \aleph_\mu)$, hence we know that pcf(a) has a bound $< \aleph_\gamma$ (and $|a| \leqq \sigma$) so there are $\theta(\gamma, 1), \ldots, \theta(\gamma, \eta)$, $b_1^\gamma, \ldots, b_\eta^\gamma$ s.t.:

(i) $\theta(\gamma, l) \in a^*$,

(ii) $b_l^\gamma \subseteq a$,

(iii) $g \restriction a^* = \bigcup_{l=1}^{\eta} f^{A(\gamma),\sigma}_{\theta(\gamma,l),g(\theta(\gamma)l)} \restriction b_l^\gamma$.

So from the information in (i), (ii), (iii) we can reconstruct $g \restriction (\bigcup\{b_\beta : \beta \in w\})$, hence $g \restriction a_\gamma$, hence $g(\aleph_{\gamma+1})$. But we could easily choose the N_ζ's so that this fails (for all possible γ's).

64(2). Remark. An instance of 64 for $\aleph_\omega$ is: suppose

$$S = \left\{\delta < \omega_1 : \text{if } \mathscr{D} \text{ is an ultrafilter on } \{\aleph_{\alpha+1} : \alpha < \delta\} \text{ containing every cobounded subset then } \mathrm{cf}\left(\prod_{\alpha<\delta} \aleph_{\alpha+1}/\mathscr{D}\right) < \aleph_{\omega_1}\right\}$$

is stationary and $2^{\aleph_0} < \aleph_\omega$, then $\aleph_\omega^{\aleph_0} < \aleph_{\omega_1}$.(7) Similarly for 62.

During this proof we have really showed, e.g.,

64(3). Fact. Suppose $a \in R^{\kappa,2}$, $\lambda \in \mathrm{pcf}(a)$, $\mu = \sup(\lambda \cap \mathrm{pcf}(a))$, μ a limit cardinal, $\mathrm{cf}\,\mu > |a|$, and let $\mu = \bigcup\{\mu_i : i < \mathrm{cf}\,\mu\}$, $\langle \mu_i : i < \mathrm{cf}\,\mu\rangle$ strictly increasing continuous. Then

$$\{\delta < \mathrm{cf}\,\mu : \text{for some } \chi < \mu_\delta,\ \max[\mathrm{pcf}([\chi, \mu_\delta) \cap \mathrm{pcf}(a))] < \lambda\}$$

is not stationary

64(4). Fact. Suppose $\mu = \sup(a)$, $a \in R^{\kappa,2}$, $\theta < \kappa$, and for every $b \subseteq a$, $|b| \leqq \theta$ we have:

$(*)_b$ $\{\lambda$: for some ultrafilter $\mathscr{D}$ on b, [for $\chi < \mu (a - \chi) \in \mathscr{D} \Rightarrow \mathrm{tcf}(\Pi a, \leqq_{\mathscr{D}}) = \lambda]\}$ has cardinality $\leqq \theta$,

then $(*)_a$.

65. Proof of $(\aleph_\delta)^{\mathrm{cf}\,\delta}(\aleph_{(|\delta|^{\mathrm{cf}\,\delta})^+}$.(8) We shall rely only on §1 and §3. Let $\kappa = \mathrm{cf}\,\delta$,

(7) Just let $\mu = \kappa = \aleph_1$, $\theta = \aleph_0$ (S_2 stationary). Note that in the definition of S, we can replace "$< \aleph_{\omega_1}$" by "$< \aleph_{\gamma(\delta)}$" for some $\gamma(\delta) < \omega_1$, as the set of such cofinalities has a last element.

(8) The proof of $(\aleph_{\alpha+\delta})^{\mathrm{cf}(\delta)} < \aleph_{\alpha+(|\delta|^{\mathrm{cf}\,\delta})^+}$ when $\aleph_\alpha^{\mathrm{cf}(\delta)} < \aleph_{\alpha+\delta}$ is similar.

the case $|\delta|^\kappa \geqq \aleph_\delta$ is easy so w.l.o.g. $2^\kappa < \aleph_\delta$, $|\delta|^\kappa < \aleph_\delta$. Let $\mu \stackrel{\text{def}}{=} (|\delta|^\kappa)^+$. Let, for $a \in R^{\kappa,2}$, $a \subseteq \aleph_\delta$, $\langle\langle f^{\kappa,a}_{\lambda,\alpha} : <\lambda\rangle : \lambda \in \mathrm{pcf}(a)\rangle$ as in 16.

Let, for every limit ordinal $\alpha < \aleph_\delta$, C_α be a club of α of order type cf(α). Let M be a model with universe $\aleph_\delta$, and functions f, g s.t. for every $\alpha < \aleph_\delta$

(i) $\{\beta : \beta < \alpha\} = \{f(\alpha, i) : i < |\alpha|\}$,

(ii) for $i < |\alpha|$, $g(\alpha, f(\alpha, i)) = i$,

(iii) for $\beta < \alpha$, $f(\alpha, g(\alpha, \beta)) = \beta$,

(iv) $f(\alpha, \alpha) = |\alpha|$,

(v) $g(\alpha, \alpha) = \alpha + 1$.

Easily

$(*)_1$ if $M^* \prec M$, $Y \subseteq M^*$ and for every $\theta \geqq \aleph_0$, $[\theta, \theta^+) \cap M^* \neq \varnothing \Rightarrow \sup(Y \cap \theta^+) = \sup(M^* \cap \theta^+)$, *then* M^* is the Skolem Hull of Y in M. [Otherwise let α be minimal in M^* but not in the Skolem Hull of Y, let $\theta = |\alpha|$, so there is $\beta \in Y \cap \theta^+ \subseteq M^*$, $\alpha < \beta$ and let $i \stackrel{\text{def}}{=} g(\beta, \alpha)$, so necessarily $i \in M^*$ (as $\beta, \alpha \in M^*$), but $i < \theta \leqq \alpha$ so by α's minimality i is in the Skolem Hull of Y; but then so is $\alpha = f(\beta, i)$, contradiction.] We know that for some ordinal $\zeta(*) < (|\delta|^\kappa)^+$

$(*)_2$ for $a \in R^{\kappa,2}$, $a \subseteq [\mu^+, \aleph_\delta) \Rightarrow \max \mathrm{pcf}(a) \leqq \aleph_{\zeta(*)}$ (by 10(4)).

Let

$$F = \Big\{ f : \mathrm{Dom}\, f = a \in R^{\kappa,2} \text{ for some } \eta < \omega, \text{ pairwise disjoint } b_1, \ldots, b_n \in R^{\kappa,2} \cap [\mu^+, \aleph_\delta), \text{ and regular } \lambda_1, \ldots, \lambda_n \leqq \aleph_{\zeta(*)} \text{ and } \alpha_1 < \lambda_1, \ldots, \alpha_n < \lambda_n : \bigcup_{l=1}^{n} b_l = a,\ f = \bigcup_{l=1}^{n} (f^{\kappa,a}_{\lambda_l,\alpha_l} \restriction b_l) \Big\}.$$

Clearly

$$(*)_3 \qquad |F| \leqq \aleph_{\zeta(*)}.$$

Let $\chi = (2^{2^{\aleph_\delta}})^+$. Now for every $x \subset \aleph_\delta$, $|x| \leqq \kappa$, we can choose $\langle N^x_i : i \leqq \mu\rangle$ s.t.:

(a) $N^x_i \prec (H(\chi), \in)$,

(b) $j < i \Rightarrow N^x_j \prec N^x_i$,

(c) $\| N^x_i \| = \mu$,

(d) $\langle N^x_j : j \leqq i\rangle \in N^x_i$,

(e) $\{\alpha : \alpha \leqq \mu\} \subseteq N_0^x$,
(f) $\langle\langle f^{\kappa,a}_{\lambda,\alpha} : \alpha < \lambda\rangle : \lambda \in \mathrm{pcf}(b)$ and $b \in R^{\kappa,2} \cap [\mu^+, \aleph_\delta)$, $|b| \leqq x\rangle \in N_0$,
(g) $\langle C_\alpha : \alpha < \aleph_\delta \text{ limit}\rangle \in N_0^x$,
(h) $M \in N_0^x$,
(i) $x \in N_0^x$ (hence $x \subseteq N_0^x$).

Let for $i \leqq (|\delta|^\kappa)^+$, $g_i = g_i^x$ be a function with domain $R^{\kappa,0} \cap (\mu, \aleph_\delta)$, $g_i(\theta) = \sup(\theta \cap N_i^x)$. By 27, for every $a \in R^{\kappa,2}$, $a \subseteq (\mu, \aleph_\gamma)$ and $x \in \mathrm{pcf}(a)$ we have $g_\mu \restriction b_\lambda^a \equiv f^{a,\kappa}_{\lambda, g_\mu(\lambda)} \restriction b^a_\lambda \bmod J^{\kappa, b^a_\lambda}_{<\lambda}$. Hence (see 27)

$(*)_4$ for $a \subseteq R^{\kappa,2} \cap [\mu^+, \aleph_\delta)$, $g_\mu \restriction a \in F$ (and $a \in I^{\kappa,2}$).

It is also quite clear that for $\theta \in R^{\kappa,2} \cap (\mu, \aleph_\delta)$, $\langle g_i(\theta) : i \leqq \mu\rangle$ is strictly increasing continuous, so for i limit $\mathrm{cf}(g_i(\theta)) = \mathrm{cf}\, i > \aleph_0$ so

$$C_{x,\theta} = C_{g_\mu(\theta)} \cap \{g_i^x(\theta) : i < \mu\}$$

is a club of $g_\mu(\theta)$ and is included in N_μ^x.

Now we can define M_n^x by induction on n s.t.
(α) $M_n^x < M$, $|M_n^x| \subseteq N_\mu^x$,
(β) $\| M_n^x \| = \kappa$,
(γ) $M_n^x < M_{n+1}^x$,
(δ) $x \subseteq M_0^x$,
(ε) if $\mu < \aleph_{\alpha+1} < \aleph_\delta$, $|M_n^x| \cap [\aleph_\alpha, \aleph_{\alpha+1}) \neq \varnothing$ *then* for some $\gamma(x, \aleph_{\alpha+1}, n) \in C_{x,\aleph_{\alpha+1}} \cap M_{n+1}^x$

$$\gamma(x, \aleph_{\alpha+1}, n) > \sup(\aleph_{\alpha+1} \cap M_n^x).$$

There is no problem to carry this.

Let $a(x) = \{\aleph_{\alpha+1} : \mu < \aleph_{\alpha+1} < \aleph_\delta$, and for some n $[\aleph_\alpha, \aleph_{\alpha+1}) \cap M_n^x \neq \varnothing\}$, so $a(x) \subseteq R^{\kappa,2} \cap [\mu^+, \aleph_\delta)$ (there is no weakly inaccessible in $[\mu^+, \aleph_\delta)$!) and let

$$Y_x = \{\gamma(x, \aleph_{\alpha+1}, n) : n < \omega, [\aleph_\alpha, \aleph_{\alpha+1}) \cap M_n^x \neq 0\}.$$

Now Y_x has cardinality κ; it is a subset of $M^x \overset{\mathrm{def}}{=} \bigcup_n M_n^x$ so by $(*)_1$

$(*)_4$ $\qquad M^x$ is the Skolem Hull in M of $(M^x \cap \mu^+) \cup Y_x$.

Now

$(*)_5$ $$Y_x \subseteq \bigcup_{\theta \in a(x)} C_{g_\mu(\theta)},$$

$(*)_6$ $$\left|\bigcup_{\theta \in a(x)} C_{g_\mu(\theta)}\right| \leqq \mu, \ |Y_x| \leqq \kappa.$$

Now

$$|\{Y_x : x \subset \aleph_\delta, |x| \leqq \kappa\}|$$

$$\leqq \left| \left\{ \left\langle g_\mu^x \restriction a(x), \left\{ z : z \subseteq \bigcup_{\theta \in a(x)} C_{g_\mu(\theta)}, |z| \leqq \kappa \right\} \right\rangle : x \subseteq \aleph_\delta, |x| \leqq \kappa \right\} \right|$$

$$\leqq |F| \times \mu^\kappa \leqq |F| \times \mu \leqq \aleph_{\zeta(*)}.$$

So if $\aleph_\delta^\kappa > \aleph_{\zeta(*)}$, necessarily for some Y

$$|\{x : Y_x = Y\}| > \aleph_{\zeta(*)}$$

and easily, for some $A \subseteq \mu$,

$$|\{x : Y_x = Y, M^x \cap \mu = A\}| > \aleph_{\zeta(*)}.$$

But $x \subseteq M^x$, $\| M^x \| \leqq \kappa$, $2^\kappa < \aleph_{\zeta(*)}$; contradiction.

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